Internal double categories and 2-categories in the abelian context

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Outline

A surjective group homomorphism

$$A \xrightarrow{f} B$$

is called:

- a trivial extension if $\hat{f}$ is an isomorphism.

$$\begin{array}{ccc} [A,A] & \overset{\cong}{\underset{\hat{f}}{\dashrightarrow}} & [B,B] \\ \downarrow & & \downarrow \\ A & \underset{f}{\twoheadrightarrow} & B \end{array}$$

- a central extension if $\mathrm{Ker}(f) \subseteq \mathrm{Centr}(A)$.

$$\begin{array}{ccccc} & \mathrm{Centr}(A) & & & \\ \nearrow & & \searrow & & \\ \mathrm{Ker}(f) & \hookrightarrow & A & \underset{f}{\twoheadrightarrow} & B \end{array}$$

[Janelidze-Kelly, 1994]:

$\mathscr{C}$ (Barr-)exact category
$\mathscr{X}$ Birkhoff subcategory of $\mathscr{C}$ (replete full reflective subcategory closed under subobjects and quotients)

$$\mathscr{C} \underset{\mathsf{H}}{\overset{\mathsf{I}}{\underset{\perp}{\rightleftarrows}}} \mathscr{X}$$

$$C \rightarrowtail X$$

$$X \twoheadrightarrow C$$

Example
$\mathscr{C} = \mathrm{Grp}$, $\mathscr{X} = \mathrm{Ab}$
$\mathsf{I} = \mathrm{ab} : \mathrm{Grp} \to \mathrm{Ab}$, $B \mapsto B/[B, B]$

$B \in \mathrm{Obj}(\mathscr{C})$:

$$\mathscr{C} \underset{\mathsf{H}}{\overset{\mathsf{I}}{\rightleftarrows}} \mathscr{X} \quad (\mathsf{I} \dashv \mathsf{H}) \qquad\qquad \mathrm{Ext}_{\mathscr{C}}(B) \underset{\mathsf{H}^B}{\overset{\mathsf{I}^B}{\rightleftarrows}} \mathrm{Ext}_{\mathscr{X}}(\mathsf{I}B) \quad (\mathsf{I}^B \dashv \mathsf{H}^B)$$

$$\begin{array}{ccc} B \times_{\mathsf{HI}(B)} \mathsf{H}(X) & \twoheadrightarrow & \mathsf{H}(X) \\ {\scriptstyle \mathsf{H}^B(f)} \downarrow & \lrcorner & \downarrow {\scriptstyle \mathsf{H}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{HI}(B) \end{array}$$

$\mathscr{X}$ is called admissible if H^B is fully faithful for all $B \in \mathrm{Obj}(\mathscr{C})$.
This is always the case if $\mathscr{C}$ is a Mal'tsev category: a finitely complete category where any internal reflexive relation is an equivalence relation.

$$\mathrm{Ext}_{\mathscr{C}}(B) \;\underset{\mathsf{H}^B}{\overset{\mathsf{I}^B}{\rightleftarrows}}\; \mathrm{Ext}_{\mathscr{X}}(\mathsf{I}B) \qquad (\mathsf{I}^B \dashv \mathsf{H}^B)$$

An extension $f \in \mathrm{Ext}_{\mathscr{C}}(B)$ is called:

- trivial if it lies in the essential image of H^B.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{HI}(A) \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{HI}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{HI}(B) \end{array}$$

- central if there exists an extension g such that q_1 is trivial.

$$\begin{array}{ccc} E \times_B A & \xrightarrow{q_2} & A \\ {\scriptstyle q_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ E & \xrightarrow[g]{} & B \end{array}$$

- $\mathrm{Triv}_{\mathscr{C}}(B) \subseteq \mathrm{Centr}_{\mathscr{C}}(B) \subseteq \mathrm{Ext}_{\mathscr{C}}(B)$
- Classification of central extensions in terms of internal actions of the Galois pregroupoid

$\mathscr{C}$ finitely complete category

Internal category (groupoid) in $\mathscr{C}$:

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \overset{\curvearrowright i}{} \underset{c}{\overset{d}{\rightrightarrows}} C_0, \quad C_0 \xrightarrow{e} C_1$$

$\mathscr{A}$ abelian category

$\mathrm{Grpd}(\mathscr{A}) \cong \mathrm{Cat}(\mathscr{A}) \cong \mathrm{RG}(\mathscr{A}) \cong \mathrm{Arr}(\mathscr{A})$

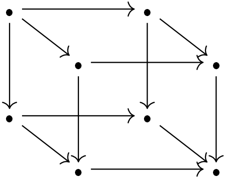
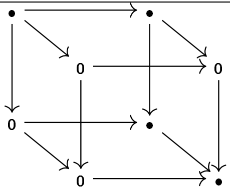
$$C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0 \ (e \colon C_0 \to C_1) \quad \mapsto \quad \mathrm{Ker}(d) \xrightarrow{\ker(d)} C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0, \quad \mathrm{cker}(d) \colon \mathrm{Ker}(d) \to C_0$$

$$A \oplus B \underset{[f, 1_B]}{\overset{\pi_2}{\rightrightarrows}} B \ (\iota_2 \colon B \to A \oplus B) \quad \leftarrow\!\shortmid \quad A \xrightarrow{f} B$$

$$\mathrm{Grpd}^2(\mathscr{A}) = \mathrm{Grpd}(\mathrm{Grpd}(\mathscr{A})) \quad \cong \quad \mathrm{Arr}^2(\mathscr{A}) = \mathrm{Arr}(\mathrm{Arr}(\mathscr{A}))$$

$$\begin{array}{ccc} C_1^1 & \overset{d^1,\ c^1}{\underset{e^1}{\rightrightarrows\!\!\!\!\leftarrow}} & C_0^1 \\ {\scriptstyle d_1}\downarrow\ \uparrow{\scriptstyle e_1}\ \downarrow{\scriptstyle c_1} & & {\scriptstyle d_0}\downarrow\ \uparrow{\scriptstyle e_0}\ \downarrow{\scriptstyle c_0} \\ C_1^0 & \overset{d^0,\ c^0}{\underset{e^0}{\rightrightarrows\!\!\!\!\leftarrow}} & C_0^0 \end{array} \qquad\qquad \begin{array}{ccc} A_1^1 & \xrightarrow{a^1} & A_0^1 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_0} \\ A_1^0 & \xrightarrow[a^0]{} & A_0^0 \end{array}$$

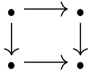
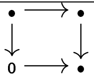
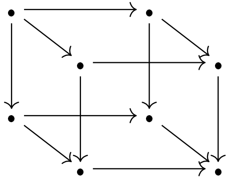
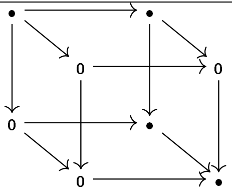
$$\frac{2\text{-}\mathrm{Grpd}(\mathscr{A})}{C_1^0 \cong C_0^0} \quad \cong \quad \frac{2\text{-}\mathrm{Arr}(\mathscr{A})}{A_1^0 = 0}$$

	$\mathrm{Arr}^n(\mathscr{A})$	$n\text{-}\mathsf{Arr}(\mathscr{A})$
$n = 2$	$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$	$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \bullet \end{array}$
$n = 3$		

$\mathbb{T}$

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

$\mathrm{Arr}^2(\mathscr{A}) = \mathscr{A}^{\mathbb{T}}$

	$\mathrm{Arr}^n(\mathscr{A})$	n-$\mathrm{Arr}(\mathscr{A})$
$n = 2$		
$n = 3$		

$\mathbb{T}$ $(\mathbb{S})$

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

$$\mathrm{Arr}^2(\mathscr{A}) = \mathscr{A}^{\mathbb{T}}$$
$$2\text{-}\mathrm{Arr}(\mathscr{A}) = \mathscr{F}, \qquad \mathrm{Obj}(\mathscr{F}) := \{F \in \mathscr{A}^{\mathbb{T}} \mid F(T) = 0 \ \forall T \notin \mathbb{S}\}$$

$\mathscr{A}$	abelian category
$\mathbb{T}$	finite category
$\mathbb{S}$	replete full subcategory of $\mathbb{T}$

$\mathscr{A}^{\mathbb{T}}$	functor category
$\mathscr{F}$	full subcategory of $\mathscr{A}^{\mathbb{T}}$ with

$$\mathrm{Obj}(\mathscr{F}) := \{F \in \mathscr{A}^{\mathbb{T}} \mid F(T) = 0 \; \forall T \notin \mathbb{S}\}$$

Proposition (E.)

$\mathscr{F}$ is an admissible Birkhoff subcategory of $\mathscr{A}^{\mathbb{T}}$.

Proof

Let $F \in \mathscr{A}^{\mathbb{T}}$. For $T \in \mathbb{T}$,

$$[T] := \{t : \tilde{T} \to T \mid \tilde{T} \notin \mathbb{S}\}$$

Let $\tau : T \to T'$ in $\mathbb{T}$. Set

$$\begin{array}{ccccc}
\bigoplus_{t\in[T]} F(d(t)) & \xrightarrow{[F(t)]_{t\in[T]}} & F(T) & \xrightarrow{\operatorname{cok}([F(t)]_t)=:(\eta_F)_T} & \operatorname{Cok}([F(t)]_t) =: \mathsf{I}(F)(T) \\
\downarrow \varphi & & \downarrow F(\tau) & & \downarrow \mathsf{I}(F)(\tau) \\
\bigoplus_{t'\in[T']} F(d(t')) & \xrightarrow[{[F(t')]_{t'\in[T']}}]{} & F(T') & \xrightarrow[\operatorname{cok}([F(t')]_{t'})=:(\eta_F)_{T'}]{} & \operatorname{Cok}([F(t')]_{t'}) =: \mathsf{I}(F)(T')
\end{array}$$

- $[F(t)]_{t\in[T]}\iota_{\hat{t}} = F(\hat{t})$ for $\hat{t} \in [T]$
- $\varphi\iota_{\hat{t}} = \iota_{\tau\hat{t}}$ for $\hat{t} \in [T]$

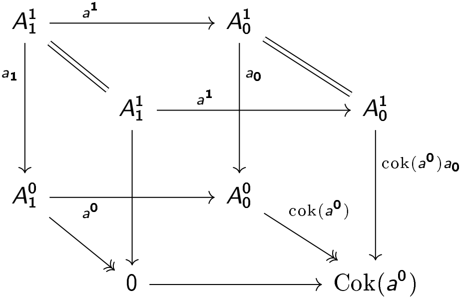
This yields the reflector $\mathsf{I} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$ with unit η.

- $\mathscr{F}$ is closed under quotients: Let $\alpha : F \Rightarrow G$ be a regular epimorphism with $F \in \mathscr{F}$. Let $T \notin \mathbb{S}$. Then

$$0 = F(T) \xrightarrow{\alpha_T} G(T)$$

  Thus, $G(T) = 0$.
- $\mathscr{F}$ is admissible: ✓

Example: $\mathsf{I} : \mathrm{Arr}^2(\mathscr{A}) \to 2\text{-}\mathrm{Arr}(\mathscr{A})$



Proposition (E.)

Let $\alpha : F \Rightarrow G$ be an extension in $\mathscr{A}^{\mathbb{T}}$. TFAE:

1. *α is trivial wrt to $\mathscr{F}$.*
2. *α_T and $(\eta_F)_T$ are jointly monic for all $T \in \mathbb{T}$.*

For $T \notin \mathbb{S}$, this means that α_T is an isomorphism.

Proof

Using [Janelidze-Kelly, 1994]

Proposition (E.)

Let $\alpha : F \Rightarrow G$ be an extension in $\mathscr{A}^{\mathbb{T}}$. TFAE:

1. α *is central wrt to* $\mathscr{F}$.
2. α_T *is an isomorphism for all* $T \notin \mathbb{S}$.

Proof

[Everaert-Gran, 2010]:

$\mathscr{C}$ semi-abelian, $\mathscr{X}$ Birkhoff subcategory, $I : \mathscr{C} \to \mathscr{X}$ protoadditive

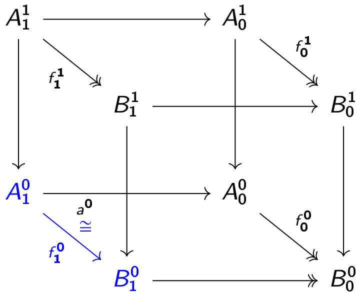
(preserves split short exact sequences)

$\Rightarrow f$ central wrt $\mathscr{X}$ iff $\mathrm{Ker}(f) \in \mathscr{X}$

Here:

$\mathrm{Ker}(\alpha) \in \mathscr{F}$ iff $Ker(\alpha_T) = 0$ for all $T \notin \mathbb{S}$

Example: Central extensions in $\mathrm{Arr}^2(\mathscr{A})$ wrt to 2-$\mathrm{Arr}(\mathscr{A})$



$\mathscr{A}^{\mathbb{T}}$

$\mathscr{F}$ full subcategory of $\mathscr{A}^{\mathbb{T}}$ with

$$\mathrm{Obj}(\mathscr{F}) := \{F \in \mathscr{A}^{\mathbb{T}} \mid F(T) = 0 \ \forall T \notin \mathbb{S}\}$$

$\mathscr{T}$ full subcategory of $\mathscr{A}^{\mathbb{T}}$ with

$$\mathrm{Obj}(\mathscr{T}) := \{F \in \mathscr{A}^{\mathbb{T}} \mid [F(t)]_{t \in [T]} \text{ epimorphism } \forall T \in \mathbb{T}\}$$

Proposition (E.)

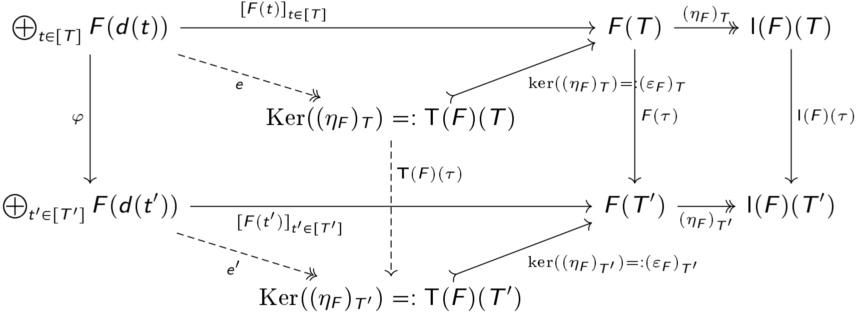
$(\mathscr{T}, \mathscr{F})$ is a torsion theory in $\mathscr{A}^{\mathbb{T}}$.

Proof

Let $F \in \mathscr{A}^{\mathbb{T}}$. Construct

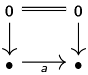
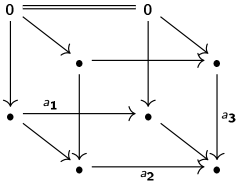
$$0 \longrightarrow \mathsf{T}(F) \xrightarrow{\varepsilon_F} F \xrightarrow{\eta_F} \mathsf{I}(F) \longrightarrow 0$$

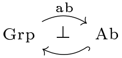
Let $\tau : T \to T'$ in $\mathbb{T}$. Consider



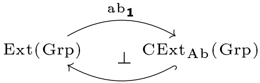
The fact that e is an epimorphism implies that $\mathsf{T}(F) \in \mathscr{T}$.

Examples

$\mathscr{A}^{\mathbb{T}}$	$\mathrm{Arr}^2(\mathscr{A})$	$\mathrm{Arr}^3(\mathscr{A})$
$\mathscr{F}$	$2\text{-}\mathrm{Arr}(\mathscr{A})$	$3\text{-}\mathrm{Arr}(\mathscr{A})$
$\mathrm{Obj}(\mathscr{T})$	 a epimorphism	 a_1 epimorphism a_2, a_3 jointly epimorphic

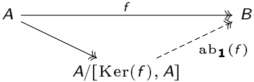


$B \mapsto B/[B, B]$

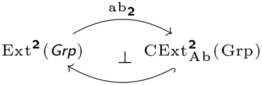


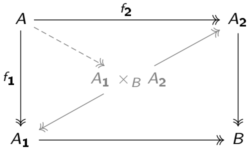
$A \xrightarrow{f} B$

$[\mathrm{Ker}(f), A] = 0$



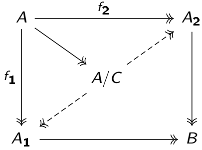
$H_2(B) = \frac{[A,A] \cap \mathrm{Ker}(f)}{[\mathrm{Ker}(f), A]}$





$[\mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2), A] = 0$

$[\mathrm{Ker}(f_1), \mathrm{Ker}(f_2)] = 0$



$C := [\mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2), A] \cdot [\mathrm{Ker}(f_1), \mathrm{Ker}(f_2)]$

$H_3(B) = \frac{[A,A] \cap \mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2)}{C}$

[Everaert, 2014]:

$$\mathscr{C} \underset{\mathsf{H}}{\overset{\mathsf{I}}{\rightleftarrows}} \mathscr{X} \quad (\mathsf{I} \dashv \mathsf{H})$$

$\mathscr{C}$ exact Mal'tsev category
$\mathscr{X}$ Birkhoff subcategory of $\mathscr{C}$
$\mathscr{E}$ class of regular epimorphisms in $\mathscr{C}$

<u>trivial extension</u>

$f \in \mathscr{E}$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{HI}(A) \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{HI}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{HI}(B) \end{array}$$

<u>central extension</u>

$f \in \mathscr{E}$ st $\exists g \in \mathscr{E}$

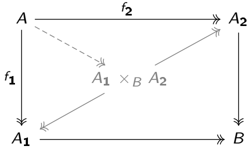
$$\begin{array}{ccc} E \times_B A & \xrightarrow{q_2} & A \\ {\scriptstyle q_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ E & \xrightarrow[g]{} & B \end{array}$$

$$\mathrm{Ext}(\mathscr{C}) \underset{\mathsf{H}_1}{\overset{\mathsf{I}_1}{\rightleftarrows}} \mathrm{CExt}_{\mathscr{X}}(\mathscr{C}) \quad (\mathsf{I}_1 \dashv \mathsf{H}_1)$$

$\mathrm{Ext}(\mathscr{C})$ full subcategory of $\mathrm{Arr}(\mathscr{C})$ corresponding to $\mathscr{E}$
$\mathrm{CExt}_{\mathscr{X}}(\mathscr{C})$ central extensions of $\mathscr{C}$ wrt $\mathscr{X}$

$$\mathrm{Ext}(\mathscr{C}) \underset{\mathsf{H}_1}{\overset{\mathsf{I}_1}{\rightleftarrows}} \mathrm{CExt}_{\mathscr{X}}(\mathscr{C}) \quad (\perp)$$

$\mathrm{Ext}(\mathscr{C})$	full subcategory of $\mathrm{Arr}(\mathscr{C})$ corresponding to $\mathscr{E}$
$\mathrm{CExt}_{\mathscr{X}}(\mathscr{C})$	central extensions of $\mathscr{C}$ wrt $\mathscr{X}$
$\mathscr{E}^1$	class of double extensions



$$\mathrm{Ext}(\mathscr{C}) \underset{\mathsf{H_1}}{\overset{\mathsf{I_1}}{\rightleftarrows}} \mathrm{CExt}_{\mathscr{X}}(\mathscr{C}) \quad (\perp)$$

$\mathrm{Ext}(\mathscr{C})$	full subcategory of $\mathrm{Arr}(\mathscr{C})$ corresponding to $\mathscr{E}$
$\mathrm{CExt}_{\mathscr{X}}(\mathscr{C})$	central extensions of $\mathscr{C}$ wrt $\mathscr{X}$
$\mathscr{E}^1$	class of double extensions

trivial double extension	central double extension
$f \in \mathscr{E}^1$	$f \in \mathscr{E}^1$ st $\exists g \in \mathscr{E}^1$

$$\begin{array}{ccc} A & \xrightarrow{\eta^1_A} & \mathsf{H_1 I_1}(A) \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{H_1 I_1}(f)} \\ B & \xrightarrow[\eta^1_B]{} & \mathsf{H_1 I_1}(B) \end{array} \qquad \begin{array}{ccc} E \times_B A & \xrightarrow{q_2} & A \\ {\scriptstyle q_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ E & \xrightarrow[g]{} & B \end{array}$$

$$\mathrm{Ext}^2(\mathscr{C}) \underset{\mathsf{H_2}}{\overset{\mathsf{I_2}}{\rightleftarrows}} \mathrm{CExt}^2_{\mathscr{X}}(\mathscr{C}) \quad (\perp)$$

$\mathrm{Ext}^2(\mathscr{C})$	full subcategory of $\mathrm{Arr}^2(\mathscr{C})$ corresponding to $\mathscr{E}^1$
$\mathrm{CExt}^2_{\mathscr{X}}(\mathscr{C})$	central extensions of $\mathrm{Ext}(\mathscr{C})$ wrt $\mathrm{CExt}_{\mathscr{X}}(\mathscr{C})$

Proposition (E.)

Let

$$\begin{array}{ccc} F & \xrightarrow{\alpha_2} & F_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow \\ F_2 & \longrightarrow & G \end{array} \qquad (\star)$$

be a double extension in $\mathscr{A}^{\mathbb{T}}$. *Tfae:*

1. $(\star)$ *is central wrt* $\mathscr{F}$.
2. $(\alpha_1)_T, (\alpha_2)_T$ *are jointly monic for all* $T \notin \mathbb{S}$.

Proof

[Everaert-Gran, 2010]:

$\mathscr{C}$ semi-abelian, $\mathscr{X}$ Birkhoff subcategory, $\mathrm{I} : \mathscr{C} \to \mathscr{X}$ protoadditive

$\Rightarrow (f_1, f_2)$ central wrt to $\mathscr{X}$

iff $\mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2) \in \mathscr{X}$

Further questions

- For $\mathscr{A}$ semi-abelian, $\mathscr{F}$ is still a Birkhoff subcategory of $\mathscr{A}^{\mathbb{T}}$ but its reflector is not necessarily protoadditive and it is not necessarily torsion-free:
  What can we say about the (higher) central extensions in this case?
- For $\mathscr{A}$ semi-abelian, we no longer have $\mathrm{Grpd}(\mathscr{A}) \cong \mathrm{Arr}(\mathscr{A})$ but $\mathrm{Grpd}(\mathscr{A}) \cong \mathrm{XMod}(\mathscr{A})$:
  How can we incorporate actions in our approach?

References

[1] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extensions*, J. Pure Appl. Algebra **97** (1994), 135–161.

[2] T. Everaert and M. Gran, *Homology of n-fold groupoids*, Theory Appl. Categ. **23** 2 (2010), 22–41.

[3] T. Everaert, *Higher central extensions in Mal'tsev categories*, Appl. Categor. Struct. **22** (2014), 961–979.

[4] T. Everaert and M. Gran, *Protoadditive functors, derived torsion theories and homology*, J. Pure Appl. Algebra **219** 8 (2015), 3629–3676.

[5] N. Egner, *Central extensions of higher internal categories in the abelian context*, in preparation.