$\mathbb{E}_n$-algebras in $(m+1)$-categories

Category Theory Octoberfest 2023

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joint work in progress with Yu Leon Liu [♣]

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Outline

- Intro and Motivations
- Stasheff's associahedron
- $\mathbb{E}_2$-algebras in an *m*-category
- Main Theorem
- Proof Sketch

Intro and Motivation

To get started, let's recall what an $(m+1)$-category [1] is

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For $m \geq -1$, a space X is m-truncated if $\pi_i(X, x) = 0$ for $i > m$ (for any basepoint $x \in X$). $m = -2$, X is -2-truncated if it is contractible.

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Example

An ordinary category is a 1-category. The category of n-truncated spaces is a $n+1$-category.

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In this talk we are interested in the following question:

Question

How do we construct $\mathbb{E}_n$-algebras in $(m+1)$-categories?

Some defintions

Before moving further, I'll recall some basic definitions

Definition

A map of operads $f : \mathcal{A} \to \mathcal{B}$ is said to be n-connected if

$$\mathrm{Mul}_{\mathcal{A}}(X_1, \cdots, X_n, Y) \to \mathrm{Mul}_{\mathcal{B}}(f(X_1), \cdots, f(X_n), f(Y))$$

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Lemma

$A \to B$ m-truncated, $\mathcal{C}$ a symmetric monoidal $(m+1)$-category

$$\mathrm{Alg}_B(\mathcal{C}) \to \mathrm{Alg}_A(\mathcal{C})$$

is an equivalence.

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Basically, moving from $\mathbb{A}_n \to \mathbb{A}_{n+1}$, we first fill in a $K_n \simeq S^{n-1} \to D^{n-1}$ cell in $\mathrm{Hom}(\mathcal{C}^{\otimes n+1}, \mathcal{C})$, and then the higher cell for units.

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Now importantly, $\mathbb{A}_n \to \mathbb{A}_{n+1}$ is $(n-3)$ connected and $\mathbb{A}_n(l) \to \mathcal{A}_{n+1}(l)$ for $l \leq n$
This implies

$$\mathrm{Alg}_{\mathbb{E}_1(\mathcal{C})} \simeq \mathrm{Alg}_{\mathbb{A}_{n+1}}(\mathcal{C})$$

where $\mathcal{C}$ is an $(m+1)$-category.

$\mathbb{E}_2$-algebras in $(m+1)$-categories

Hang on a sec, lemme grab my pen.

Main Theorem

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$$\mathbb{A}_{n_1} \otimes \mathbb{A}_{n_2} \otimes \cdots \otimes \mathbb{A}_{n_l} \to \mathbb{E}_l$$

is $(kn_1 - 2 - i)$-connected.

Consequences of Main Theorem I

Here's some consequences of this:

Corollary

$\mathbb{A}_{k_1} \otimes \mathbb{A}_{k_2} \to \mathbb{E}_2$ is

$$\begin{cases} 2k_1 - 3 & \text{when } k_1 \leq k_2 - 1 \\ k_1 + k_2 - 4 & \text{when } k_1 \geq k_2 - 1. \end{cases} \tag{1}$$

-connected. Therefore for $D+1$ category, $\mathbb{E}_2$ algebras are equivalent to

$$\begin{cases} \mathbb{A}_{d+2} \otimes \mathbb{A}_{d+2} & \text{when } D = 2d \\ \mathbb{A}_{d+2} \otimes \mathbb{A}_{d+3} & \text{when } D = 2d+1 \end{cases} \tag{2}$$

Consequences of Main Theorem II

m-category	$\mathbb{A}_{k_1} \otimes \mathbb{A}_{k_2}$
1	(2,2)
2	(2,3)
3	(3,3)
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5	(4,4)

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We see that when we go from m-category to $(m+1)$-categories, we increase one of the k_1 while "hugging the diagonal". This also gives us a way to explicitly construct $\mathbb{E}_2$ algebras by filling in the appropriate Stasheff associahedrons.

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We see that when we go from m-category to $(m+1)$-categories, we increase one of the k_1 while "hugging the diagonal". This also gives us a way to explicitly construct $\mathbb{E}_2$ algebras by filling in the appropriate Stasheff associahedrons. Furthermore, all of this generalises to $\mathbb{E}_l$.

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$$\mathrm{L}_k(\mathcal{O}_{\leq k}(n)) = \left\{ \begin{array}{c} \text{all ways to construct} \quad \text{k-ary morphisms from} \\ \leq k\text{-ary morphisms;} \end{array} \right\}$$

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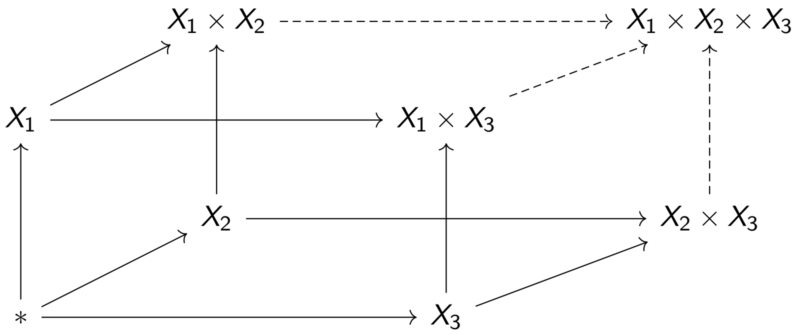
We have generally

Prop

For $\mathcal{C}$, a symmetric monoidal, unital ∞-category with colimits we have

$$R_k \mathcal{C}^{\otimes}_{\leq k}(X_1, \cdots, X_n, Y) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, f_{X_1,\cdots,X_n}\ |_{\leq k}, Y)$$

where $f_{X_1,\cdots,X_n} : \mathcal{P}(\{1, \cdots, n\}) \to \mathcal{C}$ is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$*$
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Theorem

$f : P \to Q$ is d_1-connected, $P_{\leq k} \to Q_{\leq k}$ an equivalence and R, d_2-connected then

$$P \otimes Q \to R \otimes Q$$

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R is d-connected, coherent, then

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Note that the operads $\mathbb{E}_n$ are coherent, but $\mathbb{A}_n$ are not! This is one of the complications of the proof.