What is a 2-stack?

Elena Caviglia

University of Leicester

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Sieves

Let $\mathcal{C}$ be a category with pullbacks.

Definition.

A **sieve** S on an object $C \in \mathcal{C}$ is a collection of morphisms with codomain C that is closed under precomposition with morphisms of $\mathcal{C}$.

Sieves

Let $\mathcal{C}$ be a category with pullbacks.

Definition.

A **sieve** S on an object $C \in \mathcal{C}$ is a collection of morphisms with codomain C that is closed under precomposition with morphisms of $\mathcal{C}$.

The sieve S can also be seen as a subfunctor of $y(C)$, i.e. a natural transformation

$$S \colon R_S \Rightarrow y(C)$$

with injective components.

Here R_S assigns to every object $D \in \mathcal{C}$ the set of arrows in S with domain D and it is defined on morphisms by precomposition.

Grothendieck topology on a category

Definition.

A **Grothendieck topology** τ on $\mathcal{C}$ is an assigment for each object $C \in \mathcal{C}$ of a collection $\tau(C)$ of sieves on C, called **covering sieves**, in a way such that

(T0) the maximal sieve $y(C)$ is in $\tau(C)$;

(T1) if $S \in \tau(C)$, then for every arrow $f \colon D \to C$ we have that $f^*S \in \tau(D)$;

(T2) if $S \in \tau(C)$ and R is a sieve on C such that for every $f \colon D \to C$ in S we have that $f^*R \in \tau(D)$, then $R \in \tau(C)$.

Descent data

Definition.

Let $F\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ be a pseudofunctor and let S be a sieve on $C \in \mathcal{C}$. A **descent datum on S for F** is an assignment for every morphism $D \xrightarrow{f} C$ in S of an object $W_f \in F(D)$ and, for every pair of composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, of an isomorphism $\varphi^{f,g}\colon g^* W_f \xrightarrow{\simeq} W_{f\circ g}$ such that, given morphisms $D'' \xrightarrow{h} D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} h^*(g^* W_f) & \xrightarrow{h^*\varphi^{f,g}} & h^*(W_{f\circ g}) \\ \big\downarrow \wr & & \big\downarrow \varphi^{f\circ g,h} \\ (g\circ h)^*(W_f) & \xrightarrow[\varphi^{f,g\circ h}]{} & W_{f\circ g\circ h}. \end{array}$$

Descent data

Definition.

This descent datum is called **effective** if there exist an object $W \in F(C)$ and, for every morphism $D \xrightarrow{f} C \in S$, an isomorphism

$$\psi^f \colon f^*(W) \xrightarrow{\simeq} W_f$$

such that, given morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} g^*(f^*(W)) & \xrightarrow{g^*\psi^f} & g^*(W_f) \\ \downarrow \wr & & \downarrow \varphi^{f,g} \\ (f \circ g)^* W & \xrightarrow{\psi^{f \circ g}} & W_{f \circ g}. \end{array}$$

Definition of stack

Definition.

A pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ is a **stack** if it satisfies the following conditions:

- Every descent datum for F is effective;
- (*Gluing of morphisms*) Given a covering sieve S on C, objects X and Y of $F(C)$ and for every $f\colon D \to C$ in S a morphism $\varphi_f\colon f^*X \to f^*Y$ in $F(D)$ such that $g^*(\varphi_f) = \varphi_{f\circ g}$, there exists a morphism $\eta\colon X \to Y$ such that $f^*\eta = \varphi_f$;
- (*Uniqueness of gluings*) Given a covering sieve S on C, objects X and Y of $F(C)$ and morphisms $\varphi, \psi : X \to Y$ such that for every $f\colon D \to C$ in S $f^*\varphi = f^*\psi$, then $\varphi = \psi$.

A useful characterization

Proposition (Street).

Let $F\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ *be a pseudofunctor. The following facts are equivalent:*

(1) F *is a stack;*

(2) for every object $C \in \mathcal{C}$ *and every covering sieve* $S\colon R_S \Rightarrow y(C)$ *in* $\tau(C)$ *the functor*

$$- \circ S\colon [\mathcal{C}^{\text{op}}, \mathcal{C}at](y(C), F) \longrightarrow [\mathcal{C}^{\text{op}}, \mathcal{C}at](R_S, F)$$

is an equivalence of categories.

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is an equivalence of categories.

- essentially surjective $\approx$ every descent datum is effective
- full $\approx$ gluing of morphisms
- faithful $\approx$ uniqueness of gluings

Bisieves

Let $\mathcal{K}$ be a small 2-category with bi-iso-comma objects.

Definition (Street).

A **bisieve** S over $C \in \mathcal{K}$ is a fully faithful arrow $S\colon R \Rightarrow y(C)$ in $\mathcal{B}icat(\mathcal{K}^{\mathrm{op}}, \mathcal{C}at)$.

- since S is a pseudonatural transformation, the bisieve is close under precomposition only up to isomorphism;
- for our purposes we can consider bisieves that are injective on objects.

Grothendieck topology on a 2-category

Definition (Street).

A **bitopology** τ on $\mathcal{K}$ is an assignment for each object $\mathcal{C} \in \mathcal{K}$ of a collection $\tau(C)$ of bisieves on C, called **covering bisieves**, in a way such that

(T0) the identity of $y(C)$ is in $\tau(C)$;

(T1) for all $S\colon R \to y(C)$ in $\tau(\mathcal{Y})$ and all arrows $f\colon D \to C$ in $\mathcal{K}$, the bi-iso-comma object

$$\begin{array}{ccc} P & \longrightarrow & y(D) \\ \downarrow & \overset{\sim}{\swarrow} & \downarrow{\scriptstyle -\circ f} \\ R & \underset{S}{\longrightarrow} & y(C) \end{array}$$

has the top arrow is in $\tau(D)$;

(T2) being a bisieve in τ can be checked locally.

Definition (C.).

Let $(\mathcal{K}, \tau)$ be a bisite. A trihomomorphism $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ is a **2-stack** if for every object $C \in \mathcal{K}$ and every bisieve $S\colon R \Rightarrow y(C)$ in $\tau(C)$ the pseudofunctor

$$- \circ S\colon \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(S, F)$$

is a biequivalence.

The tricategorical Yoneda lemma

Theorem (Buhné).

Let $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ be a trihomomorphism. For every $C \in \mathcal{K}$ there exists a biequivalence

$$\Gamma\colon F(C) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F)$$

which is natural in C.

- an object $X \in F(C)$ corresponds to a tritransformation $\sigma\colon y(C) \Rightarrow F$ of component $\sigma_D\colon y(D) \to F(D)$ that sends $E \xrightarrow{f} D$ to $F(f)(X)$;
- a morphism $\psi\colon X \to Y$ in $F(C)$ corresponds to a trimodification;
- a 2-cell in $F(C)$ corresponds to a perturbation;

2-stacks: an equivalent definition

Definition (C.).

Let $(\mathcal{K}, \tau)$ be a bisite. A trihomomorphism $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ is a **2-stack** if for every object $\mathcal{C} \in \mathcal{K}$ and every bisieve $S\colon R \Rightarrow \mathcal{K}(-, \mathcal{C})$ in $\tau(\mathcal{C})$ the pseudofunctor

$$(- \circ S) \circ \Gamma\colon F(C) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(S, F)$$

is a biequivalence.

2-stacks: an equivalent definition

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$$(- \circ S) \circ \Gamma \colon F(C) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\text{op}}, \mathcal{B}icat)(S, F)$$

is a biequivalence.

Remark.

Assuming the axiom of choice, biequivalence means:

(1) surjective on equivalence classes of objects;

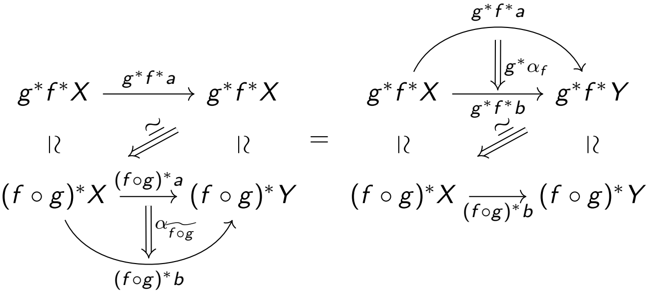
(2) essentially surjective on morphisms;

(3) fully-faithful on 2-cells.

Full on 2-cells

The fullness on 2-cells yields a condition of existence of gluings on 2-cells.

Given two morphisms $a, b\colon X \to Y$ in $F(C)$ and for every $D \xrightarrow{f} C \in S$ 2-cells $\alpha_f\colon f^*a \Rightarrow f^*b$ such that for every morphism $E \xrightarrow{g} D$ in $\mathcal{K}$ we have



there exists a 2-cell $\alpha\colon a \Rightarrow b$ in $F(C)$ such that $f^*\alpha = \alpha_f$ for every $D \xrightarrow{f} C \in S$.

Faithful on 2-cells

The faithfulness on 2-cells yields a condition of uniqueness of gluings on 2-cells.

Given 2-cells

$$X \underset{b}{\overset{a}{\underset{\Downarrow \alpha}{\rightrightarrows}}} Y \quad \text{and} \quad X \underset{b}{\overset{a}{\underset{\Downarrow \beta}{\rightrightarrows}}} Y$$

in $F(C)$ such that $f^*\alpha = f^*\beta$ for every $D \xrightarrow{f} C \in S$, we have $\alpha = \beta$.

Essentially surjective on morphisms

The essential surjectivity on morphisms yields the condition that
every descent datum on morphisms is effective.

A **descent datum on morphisms** on S for F is an assignment for every morphism $D \xrightarrow{f} C$ in S of a morphism

$$w_f \colon f^*X \longrightarrow f^*Y$$

in $F(D)$ and, for every pair of composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, of an invertible 2-cell

$$\begin{array}{ccc}
g^*f^*X & \xrightarrow{\quad g^*w_f \quad} & g^*f^*Y \\
\wr\| & & \wr\| \\
(f \circ g)^*X & \overset{\simeq}{\underset{\varphi^{f,g}}{\Swarrow}} & (f \circ g)^*Y \\
\wr\| & & \wr\| \\
(\widetilde{f \circ g})^*X & \xrightarrow[\quad w_{\widetilde{f \circ g}} \quad]{} & (\widetilde{f \circ g})^*Y
\end{array}$$

Essentially surjective on morphisms

Moreover, for every 2-cell $\gamma\colon f \Rightarrow f'$ with $f, f'\colon D \to C$ in S, we have an invertible 2-cell in $F(D)$

$$\begin{array}{ccc} f^*X & \xrightarrow{w_f} & f^*Y \\ {\scriptstyle \gamma^*_X}\downarrow & \overset{\sim}{\swarrow}{\scriptstyle \eta_\gamma} & \downarrow{\scriptstyle \gamma^*_Y} \\ f'^*X & \xrightarrow[w_{f'}]{} & f'^*Y \end{array}$$

and these 2-cells satisfy some compatibility conditions.

Essentially surjective on morphisms

Moreover, for every 2-cell $\gamma \colon f \Rightarrow f'$ with $f, f' \colon D \to C$ in S, we have an invertible 2-cell in $F(D)$

$$\begin{array}{ccc} f^*X & \xrightarrow{\ w_f\ } & f^*Y \\ {\scriptstyle \gamma^*_X}\big\downarrow & \overset{\eta_\gamma}{\underset{\simeq}{\swarrow}} & \big\downarrow{\scriptstyle \gamma^*_Y} \\ f'^*X & \xrightarrow[\ w_{f'}\ ]{} & f'^*Y \end{array}$$

and these 2-cells satisfy some compatibility conditions.

Given morphisms $D'' \xrightarrow{h} D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$ the given 2-cells need to satisfy the **cocycle condition on morphisms**, that relates the 2-cells $h^*\varphi^{f,g}$, $\widetilde{\varphi^{f \circ g,h}}$ and $\varphi^{f,g\circ h}$.

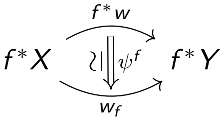
We also need to ask a condition involving the descent along the identity.

Essentially surjective on morphisms

This descent datum on morphisms is **effective** if there exist a morphism

$$w\colon X \longrightarrow Y$$

in $F(C)$ and, for every morphism $D \xrightarrow{f} C \in S$, an invertible 2-cell



such that the choice of the 2-cells is compatible w. r. to 2-cells between parallel morphisms and given morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, we have

$$\psi^{\widetilde{f\circ g}} \circ \varphi^{\widetilde{f\circ g}} = \varepsilon \circ g^*\psi^f.$$

Surjective on equivalence classes of objects

The surjectivity up to eq. classes of objects yields the condition that every *weak* descent datum (on objects) is *weakly* effective.

A **weak descent datum** on S for F is an assignment for every morphism $D \xrightarrow{f} C$ in S of an object $W_f \in F(D)$ and for every pair of composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, of an equivalence

$$\varphi^{f,g} \colon W_{\widetilde{f \circ g}} \xrightarrow{\sim} g^* W_f .$$

Moreover, we have an assigment for every 2-cell $\gamma \colon f \Rightarrow f'$ with $f, f' \colon D \to C$ in S of a morphism

$$w_\gamma \colon W_f \longrightarrow W_{f'}$$

in a pseudofunctorial way.

Surjective on equivalence classes of objects

Given morphisms $D'' \xrightarrow{h} D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the given equivalences need to satisfy the **weak cocycle condition**, i.e. there exists an invertible 2-cell

$$\begin{array}{ccccc}
W_{\widetilde{\widetilde{f\circ g}\circ h}} & \xrightarrow{\varphi^{\widetilde{f\circ g},h}} & h^* W_{\widetilde{f\circ g}} & \xrightarrow{h^*\varphi^{f,g}} & h^* g^* W_f \\
\wr\| & & \swarrow\!\!\swarrow \; \simeq & & \wr\| \\
W_{\widetilde{f\circ g\circ h}} & & \xrightarrow[\varphi^{f,g\circ h}]{} & & (g\circ h)^* W_f
\end{array}$$

We also need to ask a condition involving the descent along the identity and additional coherence conditions.

Surjective on equivalence classes of objects

This descent datum is **weakly effective** if there exist an object $W \in F(C)$ and, for every morphism $D \xrightarrow{f} C \in S$, an equivalence

$$\psi^f \colon W_f \xrightarrow{\sim} f^*(W)$$

such that, given morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, there exists an invertible 2-cell

$$\begin{array}{ccccc} W_{\widetilde{f\circ g}} & \xrightarrow{\varphi^{f,g}} & g^* W_f & \xrightarrow{g^*\psi^f} & g^* f^* W \\ \Vert & & \overset{\simeq}{\Longleftarrow} & & \wr\Vert \\ W_{\widetilde{f\circ g}} & & \xrightarrow[\psi^{\widetilde{f\circ g}}]{} & & (f\circ g)^* W. \end{array}$$

We also need to ask additional coherence conditions.

Motivating example: quotient 2-stacks

Definition (C.).

The **quotient pre-2-stack** $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{G}ray$ is defined as follows:

- for every object $\mathcal{Y} \in \mathcal{K}$ we define $[\mathcal{X}/\mathcal{G}](\mathcal{Y})$ as the 2-category of pairs $(\mathcal{P}, \alpha)$ where $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$ is a principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$ and $\alpha\colon \mathcal{P} \to \mathcal{X}$ is a $\mathcal{G}$-equivariant morphism;
- for every morphism $f\colon \mathcal{Z} \to \mathcal{Y}$ in $\mathcal{K}$, we define the 2-functor
$$[\mathcal{X}/\mathcal{G}](f) = f^*\colon [\mathcal{X}/\mathcal{G}](\mathcal{Y}) \to [\mathcal{X}/\mathcal{G}](\mathcal{Z})$$
via iso-comma object along f;
- for every 2-cell $\Lambda\colon f \Rightarrow g\colon \mathcal{Z} \to \mathcal{Y}$, we define $[\mathcal{X}/\mathcal{G}](\Lambda)$ using the universal property of the iso-comma object.

Thank you for your attention!