Determinant Functors for Triangulated Categories
and
Categorical Rings

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CATEGORIFICATION OF DETERMINANTS

- $f : k^n \longrightarrow k^n$ homomorphism $\longleftrightarrow$ $n \times n$ matrix $A \in M_n(k)$

- $\Lambda^n f : \Lambda^n k \longrightarrow \Lambda^n k$

  $\omega = v_1 \wedge \cdots \wedge v_n \longmapsto \det(A)\, \omega$

  volume form

- For any finite-dim. vect. space V, and isomorphism $f : V \xrightarrow{\cong} W$

  * $\det(V) = \left(\Lambda^{\dim(V)} V, \dim(V) \right)$
  * $\det(f) = \Lambda^{\dim(V)} f$

$$\det : \mathrm{Vect}_k^{f.d.,\, iso} \longrightarrow \mathrm{Lines}_k^{\mathbb{Z}}$$

Objects $(\mathrm{Lines}_k^{\mathbb{Z}})$ $\quad (L, n)$: L 1-dim vector space $/k$

$n \in \mathbb{Z}$

Morphisms $(\mathrm{Lines}_k^{\mathbb{Z}})$ $\quad (L,n) \longrightarrow (L',n')$ isomorphism $L \xrightarrow{\simeq} L'$ if $n = n'$ and $\emptyset$ otherwise

Symmetric Structure $(\mathrm{Lines}_k^{\mathbb{Z}})$

- $(L,n) \otimes (L',n') := (L \otimes L', n+n')$
- $(L,n) \otimes (L',n') \xrightarrow{\text{comm.}} (L',n') \otimes (L,n)$, $\quad u \otimes v \longmapsto (-1)^{nn'} v \otimes u$

$\mathrm{Lines}_k^{\mathbb{Z}}$ is a Picard Groupoid (= symm. categorical group) and

$$\det : \mathrm{Vect}_k^{\text{f.d.}, \, iso} \longrightarrow \mathrm{Lines}_k^{\mathbb{Z}}$$

Categorifies the determinant

Determinant Functors (Deligne, Le déterminant de la cohomologie '87)

- $\mathcal{E}$: exact category (w/ short exact sequences)
- $\mathcal{P}$: Picard Groupoid
- A determinant is a functor

$$\det : \operatorname{iso} \mathcal{E} \longrightarrow \mathcal{P}$$

equipped with additivity data

$$\det(\Delta) : \det(Z) + \det(X) \longrightarrow \det(Y)$$

for each exact sequence $\Delta : X \rightarrowtail Y \twoheadrightarrow Z$ satisfying

naturality

associativity

commutativity

Triangulated Categories

- Additive category $\mathcal{T}$
- Equivalence $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ (Suspension)
- Class of distinguished triangles $\Delta : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma$

  $gf = hg = 0$

  satisfying well known axioms

- A functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ between triangulated categories is exact if:

  — $F\Sigma \simeq \Sigma F$

  — $F(\Delta) : F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma F(X)$ is distinguished

DETERMINANT FUNCTORS (M. Breuning '06 – '11)

- $\mathcal{T}$: Triangulated category
- $\mathcal{P}$: Picard Groupoid
- A determinant is a functor

$$\det : \operatorname{iso}\mathcal{T} \longrightarrow \mathcal{P}$$

equipped with additivity data

$$\det(\Delta) : \det(Z) + \det(X) \longrightarrow \det(Y)$$

for each distinguished triangle

$$\Delta : \quad X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

satisfying naturality, associativity commutativity

Naturality:

$$
\begin{tikzcd}
\Delta: \arrow[d, "\Phi"'] & X \arrow[r] \arrow[d, "\cong"] & Y \arrow[r] \arrow[d, "\cong"] & Z \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\cong"] \\
\Delta': & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\det(Z) + \det(X) \arrow[r] \arrow[d] & \det(Y) \arrow[d] \\
\det(Z') + \det(X') \arrow[r] & \det(Y')
\end{tikzcd}
$$

Commutativity:

$$\Delta_1 : \quad X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{0} \Sigma X \qquad \Delta_2 : \quad Y \longrightarrow X \oplus Y \longrightarrow X \xrightarrow{0} \Sigma X$$

$$
\rightsquigarrow \quad
\begin{tikzcd}
\det(Y) + \det(X) \arrow[rr, "\text{comm.}"] \arrow[dr] & & \det(X) + \det(Y) \arrow[dl] \\
& \det(X \otimes Y) &
\end{tikzcd}
$$

Associativity:

For every octahedron

$$
\begin{tikzcd}
 & & \Delta_3 \arrow[d, squiggly] & \Delta_4 \arrow[d, squiggly] & \\
\Delta_1 \arrow[r, squiggly] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "f'"] \arrow[d, "g"] & U \arrow[r, "f''"] & \Sigma X \arrow[d, equal] \\
\Delta_2 \arrow[r, squiggly] & X \arrow[r, "\begin{smallmatrix} h \\ \shortparallel \\ gf \end{smallmatrix}"'] & Z \arrow[r, "h'"'] \arrow[d, "g'"] & V \arrow[r, "h''"'] & \Sigma X \\
 & & W \arrow[d, "g''"] & & \\
 & & \Sigma Y & &
\end{tikzcd}
$$

Associativity:

For every octahedron

$$\begin{tikzcd}
 & & \Delta_3 \arrow[d, phantom, "\wr"] & \Delta_4 \arrow[d, phantom, "\wr"] & \\
\Delta_1 \sim & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "f'"] \arrow[d, "g"] & U \arrow[r, "f''"] \arrow[d, "t"] & \Sigma X \arrow[d, equal] \\
\Delta_2 \sim & X \arrow[r, "h"'] & Z \arrow[r, "h'"'] \arrow[d, "g'"] & V \arrow[r, "h''"'] \arrow[d, "t'"] & \Sigma X \arrow[d, "\Sigma f"] \\
 & & W \arrow[r, equal] \arrow[d, "g''"] & W \arrow[r, "g''"'] \arrow[d, "t''"] & \Sigma Y \\
 & & \Sigma Y \arrow[r, "\Sigma f"'] & \Sigma U &
\end{tikzcd}$$

Associativity:

For every octahedron

$$
\begin{tikzcd}
 & & \Delta_3 \arrow[d, squiggly] & \Delta_4 \arrow[d, squiggly] & \\
\Delta_1 \arrow[r, squiggly] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "f'"] \arrow[d, "g"] & U \arrow[r, "f''"] \arrow[d, "t"] & \Sigma X \arrow[d, equal] \\
\Delta_2 \arrow[r, squiggly] & X \arrow[r, "h"'] & Z \arrow[r, "h'"'] \arrow[d, "g'"] & V \arrow[r, "h''"'] \arrow[d, "t'"] & \Sigma X \arrow[d, "\Sigma f"] \\
 & & W \arrow[r, equal] \arrow[d, "g''"] & W \arrow[r, "g''"'] \arrow[d, "t''"] & \Sigma Y \\
 & & \Sigma Y \arrow[r, "\Sigma f"'] & \Sigma U &
\end{tikzcd}
$$

We must have:

$$
\begin{tikzcd}[column sep=small]
 & \det(Z) & \\
\det(V)+\det(X) \arrow[ur, "\det(\Delta_2)"] & & \det(W)+\det(Y) \arrow[ul, "\det(\Delta_3)"'] \\
(\det(W)+\det(U))+\det(X) \arrow[u, "\det(\Delta_4)+\mathrm{id}"] \arrow[rr, "\text{assoc.}"'] & & \det(W)+(\det(U)+\det(X)) \arrow[u, "\mathrm{id}+\det(\Delta_1)"']
\end{tikzcd}
$$

UNIVERSAL DETERMINANT

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{I} \longrightarrow \mathcal{P}$ to obtain a groupoid

$$\mathrm{DET}(\mathcal{I};\mathcal{P})$$

Theorem (Breuning '06) The 2-functor

$$\mathrm{DET}(\mathcal{I};-) : \mathrm{PIC} \longrightarrow \mathrm{GRPD}$$

is representable:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "d"] \arrow[d, "\det"'] & \mathcal{P} \\
V(\mathcal{I}) \arrow[ur, "f"'] \arrow[ur, phantom, bend left=30, "\Nearrow\alpha"] &
\end{tikzcd}$$

$\blacksquare$

Universal Determinant

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T} \longrightarrow \mathcal{P}$ to obtain a groupoid

$$\mathrm{DET}(\mathcal{T};\mathcal{P})$$

Theorem (Breuning '06) The 2-functor

$$\mathrm{DET}(\mathcal{T};-) : \mathrm{Pic} \longrightarrow \mathrm{GRPD}$$

is representable:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "d"] \arrow[d, "\det"'] & \mathcal{P} & & \mathcal{T} \arrow[r, "d"] \arrow[d, "\det"'] & \mathcal{P} \\
V(\mathcal{T}) \arrow[ur, "f"'] \arrow[u, phantom, shift left=6, "\Uparrow\alpha"] & & = & V(\mathcal{T}) \arrow[ur, "g"] \arrow[ur, bend right, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow\exists!"] \arrow[u, phantom, shift left=6, "\Uparrow\beta"] &
\end{tikzcd}$$

∎

Universal Determinant

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T} \longrightarrow \mathcal{P}$ to obtain a groupoid

$$\mathrm{DET}(\mathcal{T};\mathcal{P})$$

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is representable:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "d"] \arrow[d, "\det"'] \arrow[dr, phantom, "\Uparrow\alpha"] & \mathcal{P} \\ V(\mathcal{T}) \arrow[ur, "f"'] & \end{tikzcd} = \begin{tikzcd} \mathcal{T} \arrow[r, "d"] \arrow[d, "\det"'] \arrow[dr, phantom, "\Uparrow\beta"] & \mathcal{P} \\ V(\mathcal{T}) \arrow[ur, "g"] \arrow[ur, bend right, "f"'] \arrow[ur, phantom, bend right=20, "\Uparrow\exists!"'] & \end{tikzcd}$$

$\blacksquare$

Theorem (Muro, Tonks, Witte '08) There are natural isomorphisms with Neeman's K-Theory:

$$\pi_0 V(\mathcal{T}) \cong K_0(\mathcal{T})$$

$$\pi_1 V(\mathcal{T}) \cong K_1(\mathcal{T})$$

$\blacksquare$

Tensor Triangulated Categories (Balmer, May, Keller-Neeman, ...)

$(\mathcal{T}, \otimes, I)$ $\qquad$ $\mathcal{T}$ – triangulated

$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ biexact & (symmetric) monoidal with unit object I.

minimal requirements (Balmer), but more axioms can be considered (May, Keller-Neeman, ...)

Rmk TT-cats behave like 2-rigs, but very non canonically. Example:

$$\begin{tikzcd}
X \otimes Z \arrow[r] \arrow[d, equal] & (X \otimes Z) \oplus (Y \otimes Z) \arrow[r] & Y \otimes Z \arrow[r] \arrow[d, equal] & \Sigma(X \otimes Z) \arrow[d, equal] \\
X \otimes Z \arrow[r] & (X \oplus Y) \otimes Z \arrow[r] & Y \otimes Z \arrow[r] & \Sigma(X \otimes Z)
\end{tikzcd}$$

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X \otimes Z \arrow[r] & (X \oplus Y) \otimes Z \arrow[r] & Y \otimes Z \arrow[r] & \Sigma(X \otimes Z)
\end{tikzcd}$$

TENSOR TRIANGULATED CATEGORIES (Balmer, May, Keller-Neeman, ...)

$(\mathcal{T}, \otimes, \mathbb{I})$ $\qquad$ $\mathcal{T}$ - triangulated

minimal requirements (Balmer), but more axioms can be considered (May, Keller-Neeman, ...)

$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ biexact & (symmetric) monoidal with unit object $\mathbb{I}$.

Rmk TT-cats behave like 2-rigs, but very non canonically. Example:

$$\begin{tikzcd}
X \otimes Z \arrow[r] \arrow[d, equal] & (X \otimes Z) \oplus (Y \otimes Z) \arrow[r] \arrow[d, "\cong\ \exists"] & Y \otimes Z \arrow[r] \arrow[d, equal] & \Sigma(X \otimes Z) \arrow[d, equal] \\
X \otimes Z \arrow[r] & (X \oplus Y) \otimes Z \arrow[r] & Y \otimes Z \arrow[r] & \Sigma(X \otimes Z)
\end{tikzcd}$$

non canonical distributor

Similarly for other "structural" map.

Tensor Triangulated Categories (Balmer, May, Keller-Neeman, ...)

$(\mathcal{T}, \otimes, I)$ $\quad \mathcal{T}$ - triangulated

minimal requirements (Balmer), but

$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ biexact & (symmetric) monoidal with unit object I.

more axioms can be considered (May, Keller-Neeman, ...)

Rmk TT-cats behave like 2-rigs, but very non canonically. Example:

$$
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X \otimes Z \arrow[r] & (X \oplus Y) \otimes Z \arrow[r] & Y \otimes Z \arrow[r] & \Sigma(X \otimes Z)
\end{tikzcd}
$$

Question For $(\mathcal{T}, \otimes, I)$, is the universal determinant equipped with

$$V(\mathcal{T}) \times V(\mathcal{T}) \longrightarrow V(\mathcal{T}) \quad ?$$

BEHAVIOR WITH RESPECT TO BIEXACT (n-EXACT) FUNCTORS

- $F : \mathcal{T}_1 \times \mathcal{T}_2 \longrightarrow \mathcal{T}$ is biexact if (1) exact in each variable

  (2)
$$\begin{array}{ccc} F(\Sigma X, \Sigma Y) & \longrightarrow & \Sigma F(X, \Sigma Y) \\ \downarrow & (-1) & \downarrow \\ \Sigma F(\Sigma X, Y) & \longrightarrow & \Sigma^2 F(X, Y) \end{array}$$

- Same for n-exact $F : \mathcal{T}_1 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{T}$

- GRPD-enriched multicategory (= 2-multicategory) $\underline{\underline{\mathrm{TRCAT}}}$ with

$$\underline{\underline{\mathrm{TRCAT}}}(\mathcal{T}_1, \dots, \mathcal{T}_n; \mathcal{T}) = \text{n-exact functors } \mathcal{T}_1 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{T} \text{ and natural isomorphisms}$$

(Schnürer, '15)

- Same for PIC : multicategory (GRPD-enriched) of PICARD Groupoid $\underline{\underline{\mathrm{Pic}}}$ with

$$\underline{\underline{\mathrm{Pic}}}(\mathcal{P}_1, \dots, \mathcal{P}_n; \mathcal{P}) = \text{n-monoidal functors \& nat. isomorphisms}$$

- $V(\mathcal{T}_1) \times \cdots \times V(\mathcal{T}_n) \longrightarrow V(\mathcal{T})$ in $\underline{\underline{\mathrm{Pic}}}$

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_n$ — triangulated cats

$\mathcal{P}$ — Picard Groupoid

- n-functor $\det : \mathrm{iso}\,\mathcal{T}_1 \times \dots \times \mathrm{iso}\,\mathcal{T}_n \longrightarrow \mathcal{P}$
- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i = 1, \dots, n$

MULTIDETERMINANTS

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- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i = 1, \dots, n$
- Compatibility with $f_i : X_i \to X_i' \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_n$ — triangulated cats

$\mathcal{P}$ — Picard Groupoid

- n-functor $\det : \mathrm{iso}\,\mathcal{T}_1 \times \dots \times \mathrm{iso}\,\mathcal{T}_n \longrightarrow \mathcal{P}$
- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i = 1, \dots, n$
- $\Delta_i : X_i \to Y_i \longrightarrow Z_i \to \Sigma X_i$ in $\mathcal{T}_i$

  $\Delta_j : X_j \to Y_j \longrightarrow Z_j \to \Sigma X_j$ in $\mathcal{T}_j$

- Compatibility with $f_i : X_i \to X_i' \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

Multideterminants

$\mathcal{T}_1, \dots, \mathcal{T}_n$ — triangulated Cats

$\mathcal{P}$ — Picard Groupoid

- n-functor $\det : \mathrm{iso}\,\mathcal{T}_1 \times \dots \times \mathrm{iso}\,\mathcal{T}_n \longrightarrow \mathcal{P}$
- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i = 1, \dots, n$
- $\Delta_i : X_i \to Y_i \to Z_i \to \Sigma X_i$ in $\mathcal{T}_i$

  $\Delta_j : X_j \to Y_j \to Z_j \to \Sigma X_j$ in $\mathcal{T}_j$

Notation: $\det(X_1, \dots, X_n) = [X_1, \dots, X_n]$

$$\begin{tikzcd}
& {[Z_i, Y_j] + [X_i, Y_j]} \arrow[dl] & \left([Z_i, Z_j] + [Z_i, X_j]\right) + \left([X_i, Z_j] + [X_i, X_j]\right) \arrow[l] \arrow[d, "\text{comm + assoc}"] \\
{[Y_i, Y_j]} & \circlearrowright & \\
& {[Y_i, Z_j] + [Y_i, X_j]} \arrow[ul] & \left([Z_i, Z_j] + [X_i, Z_j]\right) + \left([Z_i, X_j] + [X_i, X_j]\right) \arrow[l]
\end{tikzcd}$$

- Compatibility with $f_i : X_i \to X_i' \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

Universal MultiDeterminant

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{P}$ to obtain a groupoid $\mathrm{DET}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n ; \mathcal{P})$.

Recall:

(symm.) multicategory $\underline{\mathcal{M}}$ $\rightsquigarrow$ / $\leftsquigarrow$ (symm.) monoidal category $\underline{\underline{\mathcal{M}}}^{\times}$

Universal MultiDeterminant

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{P}$

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Recall:

(symm.) multicategory $\underline{\mathcal{M}}$ $\rightsquigarrow$ / $\leftsquigarrow$ (symm.) monoidal category $\underline{\underline{\mathcal{M}}}^{\times}$

Theorem (E.A., C. Lester) There is an equivalence of groupoids

$$\mathrm{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n ; \mathcal{P}) \simeq \underline{\underline{\mathrm{Pic}}}\,(V(\mathcal{T}_1), \dots, V(\mathcal{T}_n) ; \mathcal{P})$$

That is, the object $(V(\mathcal{T}_1), \dots, V(\mathcal{T}_n))$ of $\underline{\underline{\mathrm{Pic}}}^{\times}$ corepresents:

$$\mathrm{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n ; -) : \underline{\underline{\mathrm{Pic}}}^{\times} \longrightarrow \mathrm{GRPD} \qquad \blacksquare$$

UNIVERSAL MULTIDETERMINANT

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{P}$

to obtain a groupoid $\mathrm{DET}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n ; \mathcal{P})$.

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That is, the object $(V(\mathcal{T}_1), \ldots, V(\mathcal{T}_n))$ of $\underline{\underline{\mathrm{PIC}}}^{\times}$ corepresents:

$$\mathrm{DET}(\mathcal{T}_1, \ldots, \mathcal{T}_n ; -) : \underline{\underline{\mathrm{PIC}}}^{\times} \longrightarrow \mathrm{GRPD}$$

■

Theorem (E.A., C.LESTER) For each Picard groupoid $\mathcal{P}$ $(*)$ determines

$$\mathrm{DET}(- ; \mathcal{P}) : \underline{\underline{\mathrm{TRCAT}}}^{\times\, op}_{\mathrm{Verdier}} \longrightarrow \mathrm{GRPD}$$

■

Verdier Structures

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & \Sigma X'' \arrow[d] \\
\Sigma X' \arrow[r] & \Sigma Y' \arrow[r] & \Sigma Z' \arrow[r] & \Sigma^2 X'
\end{tikzcd}
$$

The 9-diagram:

Each line is a distinguished triangle.

Verdier Structures

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

$$
\begin{tikzcd}
X' \arrow[rr] \arrow[d] \arrow[drr] & & Y' \arrow[rr] \arrow[d] & & Z' \arrow[rr] \arrow[d] & & \Sigma X' \arrow[d] \\
X \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] \arrow[dr] & & Z \arrow[rr] \arrow[dd] & & \Sigma X \arrow[dd] \\
& & & A & & & \\
X'' \arrow[rr] \arrow[d] & & Y'' \arrow[rr] \arrow[d] & & Z'' \arrow[rr] \arrow[d] \arrow[drr, phantom, "-1"] & & \Sigma X'' \arrow[d] \\
\Sigma X' \arrow[rr] & & \Sigma Y' \arrow[rr] & & \Sigma Z' \arrow[rr] & & \Sigma^2 X'
\end{tikzcd}
$$

The 9-diagram:

Each line is a distinguished triangle.

Consider the triangle

$$X' \longrightarrow Y \longrightarrow A \longrightarrow \Sigma X'$$

Verdier Structures

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

The 9-diagram has a Verdier structure if there exist octahedra

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & \Sigma X'' \arrow[d] \\
\Sigma X' \arrow[r] & \Sigma Y' \arrow[r] & \Sigma Z' \arrow[r] & \Sigma^2 X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d, equal] \\
X' \arrow[r] & Y \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \Sigma X' \\
& Y'' \arrow[r, equal] \arrow[d] & Y'' \arrow[d] & \\
& \Sigma Y' \arrow[r] & \Sigma Z' &
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & & \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d] \\
X'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & Z \arrow[r] & \Sigma X'' \\
\Sigma X' \arrow[r, equal] & \Sigma X' & &
\end{tikzcd}
$$

(1)

$$
\begin{tikzcd}
X'' \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X'' \arrow[d] \\
X'' \arrow[r] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & \Sigma X'' \\
& \Sigma Z' \arrow[r, equal] \arrow[d] & \Sigma Z' \arrow[d] & \\
& \Sigma A \arrow[r] & \Sigma Z &
\end{tikzcd}
$$

(3)

Verdier Structures (E.A., C. Lester)

$F : \mathcal{T}_1 \times \cdots \times \mathcal{T}_n \longrightarrow \mathcal{T}$ multiexact admits a Verdier structure if for all $\Delta_i \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$, $i < j$, the diagram

$$\begin{tikzcd}
F(X_i, X_j) \arrow[r] \arrow[d] & F(Y_i, X_j) \arrow[r] \arrow[d] & F(Z_i, X_j) \arrow[r] \arrow[d] & \Sigma F(X_i, X_j) \arrow[d] \\
F(X_i, Y_j) \arrow[r] \arrow[d] & F(Y_i, Y_j) \arrow[r] \arrow[d] & F(Z_i, Y_j) \arrow[r] \arrow[d] & \Sigma F(X_i, Y_j) \arrow[d] \\
F(X_i, Z_j) \arrow[r] \arrow[d] & F(Y_i, Z_j) \arrow[r] \arrow[d] & F(Z_i, Z_j) \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & \Sigma F(X_i, Z_j) \arrow[d] \\
\Sigma F(X_i, X_j) \arrow[r] & \Sigma F(Y_i, X_j) \arrow[r] & \Sigma F(Z_i, X_j) \arrow[r] & \Sigma^2 F(X_i, X_j)
\end{tikzcd}$$

admits a Verdier structure

CATEGORICAL RINGS

Def A Picard groupoid $(\mathcal{P}, +, 0)$ is a categorical ring if there exists a second monoidal structure

$$\bullet : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$$

which is biexact, unital, associative.

This is the biased version: associativity, etc. require higher arity structures by composition. Alternatively:

Def A categorical ring is a (commutative) monoid in $\underline{\underline{\mathrm{Pic}}}$

This is the unbiased version.

Rmk Similarly, we consider a TT-cat $(\mathcal{T}, \otimes, \mathbb{I})$ as a monoid in $\underline{\underline{\mathrm{TRCAT}}}$

DET & $\otimes$

Theorem (E. A., C. Lester)

- $(\mathcal{T}, \otimes, I)$ - Tensor Triangulated cat.
- $\otimes$ admits Verdier

Then the universal Picard groupoid $V(\mathcal{T})$ is a categorical ring

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Idea of proof

$$\begin{tikzcd}
\mathcal{T} \times \mathcal{T} \arrow[rr, "\otimes"] \arrow[d, "\det \times \det"'] \arrow[drr, "\det \circ \otimes"] & & \mathcal{T} \arrow[d, "\det"] \\
V(\mathcal{T}) \times V(\mathcal{T}) \arrow[rr, "\odot"'] \arrow[ur, phantom, "\Nearrow"] & & V(\mathcal{T})
\end{tikzcd}$$

∎

DET & ⊗

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Idea of proof

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V(\mathcal{T}) \times V(\mathcal{T}) \arrow[r, "\odot"'] \arrow[ur, phantom, "\Uparrow"] & V(\mathcal{T})
\end{tikzcd}$$

∎

As a Corollary, we get the well known fact

$K_0(\mathcal{T}) \cong \pi_0 V(\mathcal{T})$ ring

$K_1(\mathcal{T}) \cong \pi_1 V(\mathcal{T})$ $K_0(\mathcal{T})$-bimodule

Outlook

What does the Postnikov Invariant

$$\eta_{V(\mathcal{T})} \in THH^3(\pi_0 V(\mathcal{T}), \pi_1 V(\mathcal{T}))$$

say about $(\mathcal{T}, \otimes, \mathbb{I})$?

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THANK YOU!