

DETERMINANT FUNCTORS FOR TRIANGULATED CATEGORIES
AND
CATEGORICAL RINGS

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CATEGORIFICATION OF DETERMINANTS

• $f: k^m \rightarrow k^m$ homomorphism \leftrightarrow $n \times n$ matrix $A \in M_n(k)$

• $\wedge^m f: \wedge^m k \rightarrow \wedge^m k$

$\omega = v_1 \wedge \dots \wedge v_m \mapsto \det(A) \omega$
volume form

• For any finite-dim. vect. space V , and isomorphism $f: V \xrightarrow{\cong} W$

$$* \quad \det(V) = \left(\wedge^{\dim(V)} V, \dim(V) \right)$$

$$* \quad \det(f) = \wedge^{\dim(V)} f$$

$$\det: \text{Vect}_k^{\text{f.d., iso}} \longrightarrow \text{Limes}_k \mathbb{Z}$$

Objects $(\text{Limes}_k^{\mathbb{Z}})$

$(L, n) : L \text{ 1-dim vector space / } k$
 $n \in \mathbb{Z}$

Morphisms $(\text{Limes}_k^{\mathbb{Z}})$

$(L, n) \longrightarrow (L', n')$ isomorphism $L \xrightarrow{\cong} L'$
if $n = n'$ and \emptyset otherwise

Symmetric Structure $(\text{Limes}_k^{\mathbb{Z}})$

- $(L, n) \otimes (L', n') := (L \otimes L', n + n')$
- $(L, n) \otimes (L', n') \xrightarrow{\text{comm.}} (L', n') \otimes (L, n), \quad u \otimes v \mapsto (-1)^{nn'} v \otimes u$

$\text{Limes}_k^{\mathbb{Z}}$ is a **Picard Groupoid** (= Symm. categorical group) and

$$\det : \text{Vect}_k^{\text{f.d., iso}} \longrightarrow \text{Limes}_k^{\mathbb{Z}}$$

Categorifies the determinant

DETERMINANT FUNCTORS (DELIGNE, Le déterminant de la cohomologie '87)

- \mathcal{E} : exact category (w/ short exact sequences)
- \mathcal{P} : Picard Groupoid
- A **determinant** is a functor

$$\det : \text{iso } \mathcal{E} \longrightarrow \mathcal{P}$$

equipped with **additivity data**

$$\det(\Delta) : \det(Z) + \det(X) \longrightarrow \det(Y)$$

for each exact sequence $\Delta : X \twoheadrightarrow Y \twoheadrightarrow Z$ satisfying

naturality

associativity

commutativity

TRIANGULATED CATEGORIES

- Additive category \mathcal{T}
- Equivalence $\Sigma: \mathcal{T} \rightarrow \mathcal{T}$ (Suspension)
- Class of distinguished triangles $\Delta: X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$
satisfying well known axioms $gf = hg = 0$
- A functor $F: \mathcal{T} \rightarrow \mathcal{T}'$ between triangulated categories is exact if:

$$- F\Sigma \simeq \Sigma F$$

$$- F(\Delta): F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow \Sigma F(X)$$

is distinguished

DETERMINANT FUNCTORS (M. Breuning '06 - '11)

- \mathcal{T} : Triangulated category
- \mathcal{P} : Picard Groupoid
- A **determinant** is a functor

$$\det : \text{iso } \mathcal{T} \longrightarrow \mathcal{P}$$

equipped with **additivity data**

$$\det(\Delta) : \det(Z) + \det(X) \longrightarrow \det(Y)$$

for each **distinguished triangle**

$$\Delta : X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

satisfying **naturality, associativity, commutativity**

Naturality:

$$\begin{array}{ccccccc}
 \Delta: & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
 \Downarrow \cong & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \Delta': & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 \det(Z) + \det(X) & \longrightarrow & \det(Y) \\
 \downarrow & & \downarrow \\
 \det(Z') + \det(X') & \longrightarrow & \det(Y')
 \end{array}$$

Commutativity:

$$\Delta_1: X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{\circ} \Sigma X \qquad \Delta_2: Y \longrightarrow X \oplus Y \longrightarrow X \xrightarrow{\circ} \Sigma X$$

$$\rightsquigarrow
 \begin{array}{ccc}
 \det(Y) + \det(X) & \xrightarrow{\text{comm.}} & \det(X) + \det(Y) \\
 & \searrow & \swarrow \\
 & \det(X \oplus Y) &
 \end{array}$$

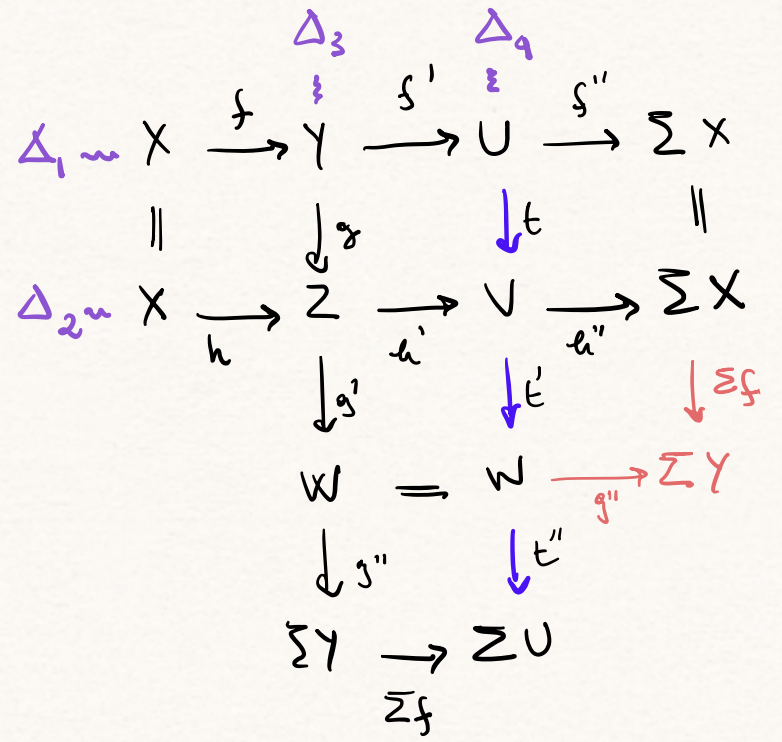
Associativity:

For every octahedron

$$\begin{array}{ccccccc}
 \Delta_1 \rightsquigarrow X & \xrightarrow{f} & Y & \xrightarrow{s'} & U & \xrightarrow{s''} & \Sigma X \\
 \parallel & & \downarrow g & & \parallel & & \parallel \\
 \Delta_2 \rightsquigarrow X & \xrightarrow{h} & Z & \xrightarrow{h'} & V & \xrightarrow{h''} & \Sigma X \\
 & \text{gf} = h & \downarrow g' & & & & \\
 & & W & & & & \\
 & & \downarrow g'' & & & & \\
 & & \Sigma Y & & & &
 \end{array}$$

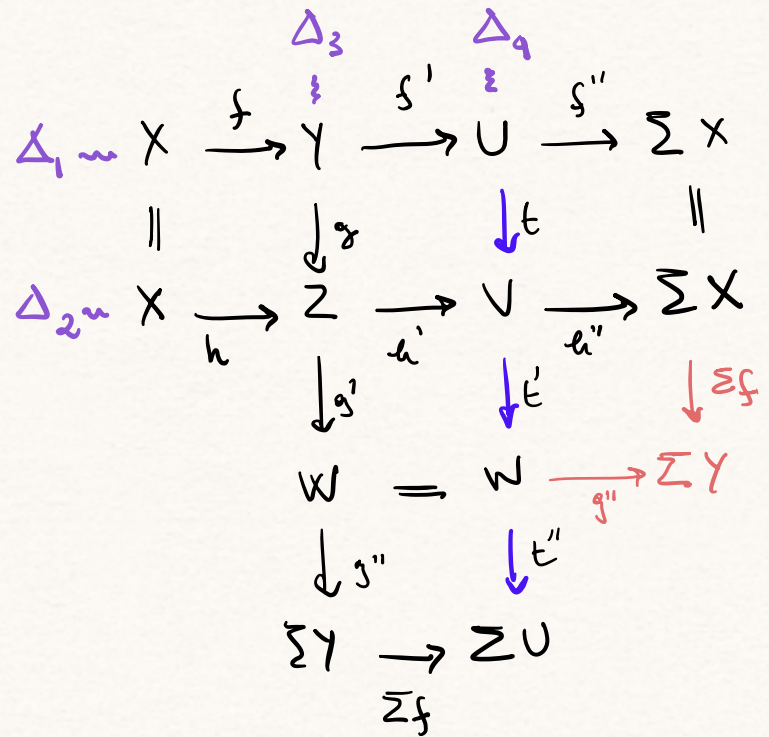
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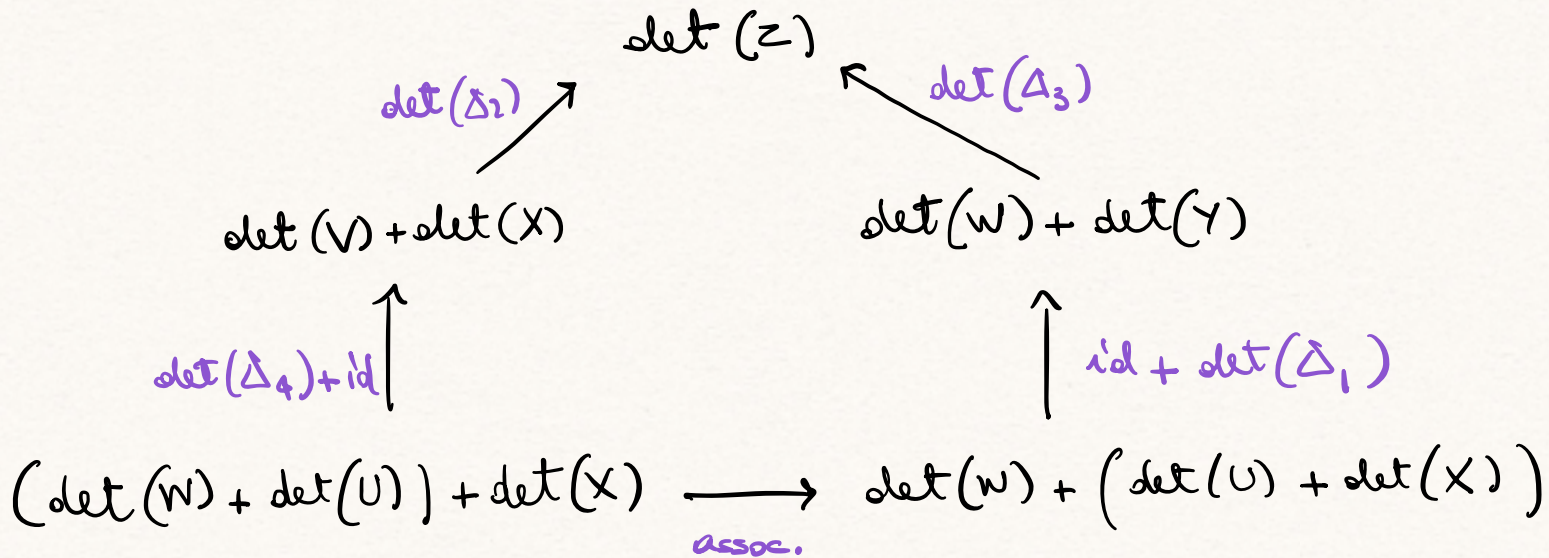


Associativity:

For every octahedron



We must have:



UNIVERSAL DETERMINANT

Define natural isomorphisms $\det \Rightarrow \det' : \mathcal{T} \longrightarrow \mathcal{P}$ to obtain a groupoid

$$\text{DET}(\mathcal{T}; \mathcal{P})$$

Theorem (Breuning '06) The 2-functor

$$\text{DET}(\mathcal{T}; -) : \text{Pic} \longrightarrow \text{GRPD}$$

is representable:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{d} & \mathcal{P} \\ \det \downarrow & \nearrow \alpha & \nearrow f \\ V(\mathcal{T}) & & \end{array}$$



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 \mathcal{T} & \xrightarrow{d} & \mathcal{P} \\
 \det \downarrow & \nearrow \alpha & \\
 V(\mathcal{T}) & \xrightarrow{f} &
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{T} & \xrightarrow{d} & \mathcal{P} \\
 \det \downarrow & \nearrow \beta & \\
 V(\mathcal{T}) & \xrightarrow{f} &
 \end{array}$$

The diagram on the right includes a red arrow γ from $V(\mathcal{T})$ to \mathcal{P} and a red arrow β' from \mathcal{P} to $V(\mathcal{T})$, with a curved blue arrow f from $V(\mathcal{T})$ to $V(\mathcal{T})$.

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 V(\mathcal{T}) & \xrightarrow{f} &
 \end{array}$$

The diagram on the right includes a red arrow β from $V(\mathcal{T})$ to \mathcal{T} , a red arrow γ from $V(\mathcal{T})$ to \mathcal{P} , and a red arrow γ' from \mathcal{P} to $V(\mathcal{T})$. A blue curved arrow f also connects $V(\mathcal{T})$ to \mathcal{P} .

Theorem (Muro, Tonks, Witte '08) There are natural isomorphisms with

Neeman's K-Theory:

$$\pi_0 V(\mathcal{T}) \cong K_0(\mathcal{T})$$

$$\pi_1 V(\mathcal{T}) \cong K_1(\mathcal{T})$$

TENSOR TRIANGULATED CATEGORIES (Balmer, May, Keller-Neeman, ...)

$(\mathcal{T}, \otimes, \mathbb{I})$ \mathcal{T} -triangulated

minimal requirements (Balmer), but

more axioms can be considered (May, Keller-Neeman, ...)

$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ birect & (symmetric) monoidal with unit object \mathbb{I} .

Remark TT-cats behave like 2-rigs, but very non canonically. Example:

$$\begin{array}{ccccccc} X \otimes Z & \longrightarrow & (X \otimes Z) \oplus (Y \otimes Z) & \longrightarrow & Y \otimes Z & \longrightarrow & \Sigma(X \otimes Z) \\ \parallel & & & & \parallel & & \parallel \\ X \otimes Z & \longrightarrow & (X \oplus Y) \otimes Z & \longrightarrow & Y \otimes Z & \longrightarrow & \Sigma(X \otimes Z) \end{array}$$

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 X \otimes Z & \longrightarrow & (X \oplus Y) \otimes Z & \longrightarrow & Y \otimes Z & \longrightarrow & \Sigma(X \otimes Z)
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 X \otimes Z & \longrightarrow & (X \oplus Y) \otimes Z & \longrightarrow & Y \otimes Z & \longrightarrow & \Sigma(X \otimes Z)
 \end{array}$$

non canonical distributor

Similarly for other "structural" map.

TENSOR TRIANGULATED CATEGORIES (Balmer, May, Keller-Neeman, ...)

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 \parallel & & \cong \downarrow \exists & & \parallel & & \parallel \\
 X \otimes Z & \longrightarrow & (X \oplus Y) \otimes Z & \longrightarrow & Y \otimes Z & \longrightarrow & \Sigma(X \otimes Z)
 \end{array}$$

Question For $(\mathcal{T}, \otimes, \mathbb{I})$, is the universal determinant equipped with

$$V(\mathcal{T}) \times V(\mathcal{T}) \longrightarrow V(\mathcal{T}) \quad ?$$

BEHAVIOR WITH RESPECT TO BIFUNCTOR (m-EXACT) FUNCTORS

- $F: \mathcal{T}_1 \times \mathcal{T}_2 \longrightarrow \mathcal{T}$ is **bifunctor** if (1) exact in each variable

$$(2) \quad F(\Sigma X, \Sigma Y) \longrightarrow \Sigma F(X, \Sigma Y)$$

$$\downarrow \quad (-1) \quad \downarrow$$

$$\Sigma F(\Sigma X, Y) \longrightarrow \Sigma^2 F(X, Y)$$

- Same for **m-exact** $F: \mathcal{T}_1 \times \dots \times \mathcal{T}_m \longrightarrow \mathcal{T}$

- GRPD-enriched **multicategory** (= 2-multicategory) TRCAT with

$$\underline{\underline{\text{TRCAT}}}(\mathcal{T}_1, \dots, \mathcal{T}_m; \mathcal{T}) = n\text{-exact functors } \mathcal{T}_1 \times \dots \times \mathcal{T}_m \longrightarrow \mathcal{T} \text{ and natural isomorphisms}$$

(Schmüser, '15)

- Same for Pic: multicategory (GRPD-enriched) of PICARD Groupoid Pic with

$$\underline{\underline{\text{Pic}}}(\mathcal{P}_1, \dots, \mathcal{P}_m; \mathcal{P}) = n\text{-monoidal functors \& mat. isomorphisms}$$

- $V(\mathcal{T}_1) \times \dots \times V(\mathcal{T}_m) \longrightarrow V(\mathcal{T})$ in Pic

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_m$ — triangulated Cets

\mathcal{P} — Picard Groupoid

- m -functor $\det : \text{iso } \mathcal{T}_1 \times \dots \times \text{iso } \mathcal{T}_m \longrightarrow \mathcal{P}$
- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i=1, \dots, m$

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_m$ — triangulated sets

\mathcal{P} — Picard Groupoid

- m -functor $\det : \text{iso } \mathcal{T}_1 \times \dots \times \text{iso } \mathcal{T}_m \longrightarrow \mathcal{P}$
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- Compatibility with $f_i: X_i \rightarrow X'_i \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_m$ — triangulated sets

\mathcal{P} — Picard Groupoid

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- $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i=1, \dots, m$
- $\Delta_i : X_i \rightarrow Y_i \rightarrow Z_i \rightarrow \Sigma X_i$ in \mathcal{T}_i
 $\Delta_j : X_j \rightarrow Y_j \rightarrow Z_j \rightarrow \Sigma X_j$ in \mathcal{T}_j
- Compatibility with $f_i : X_i \rightarrow X'_i \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

MULTIDETERMINANTS

$\mathcal{T}_1, \dots, \mathcal{T}_m$ — triangulated sets

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• n -functor $\det : \text{iso } \mathcal{T}_1 \times \dots \times \text{iso } \mathcal{T}_m \longrightarrow \mathcal{P}$

• $\det|_{\mathcal{T}_i}$ is a determinant functor in each variable $i=1, \dots, m$

• $\Delta_i : X_i \rightarrow Y_i \rightarrow Z_i \rightarrow \Sigma X_i$ in \mathcal{T}_i

Notation: $\det(X_1, \dots, X_m) = [X_1, \dots, X_m]$

$\Delta_j : X_j \rightarrow Y_j \rightarrow Z_j \rightarrow \Sigma X_j$ in \mathcal{T}_j

$$\begin{array}{ccc}
 [Z_i, Y_j] + [X_i, Y_j] & \longleftarrow & ([Z_i, Z_j] + [Z_i, X_j]) + ([X_i, Z_j] + [X_i, X_j]) \\
 \swarrow & & \downarrow \text{comm + assoc} \\
 [Y_i, Y_j] & & \\
 \nwarrow & & \\
 [Y_i, Z_j] + [Y_i, X_j] & \longleftarrow & ([Z_i, Z_j] + [X_i, Z_j]) + ([Z_i, X_j] + [X_i, X_j])
 \end{array}$$

• Compatibility with $f_i : X_i \rightarrow X'_i \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$

UNIVERSAL MULTIDETERMINANT

Define natural isomorphisms $\det \Rightarrow \det^{\vee} : \mathcal{T}_1 \times \mathcal{T}_2 \times \dots \times \mathcal{T}_n \longrightarrow \mathcal{P}$

to obtain a groupoid $\text{DET}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n; \mathcal{P})$.

Recall:

(symm.) multicategory $\underline{\mathcal{M}}$ $\begin{array}{c} \rightsquigarrow \\ \longleftarrow \end{array}$ (symm.) monoidal category $\underline{\mathcal{M}}^{\times}$

UNIVERSAL MULTIDETERMINANT

Define natural isomorphisms $\det \Rightarrow \det^3 : \mathcal{T}_1 \times \mathcal{T}_2 \times \dots \times \mathcal{T}_n \longrightarrow \mathcal{P}$
to obtain a groupoid $\text{DET}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n; \mathcal{P})$.

Recall:

(symm.) multicategory $\underline{\underline{M}} \begin{matrix} \rightsquigarrow \\ \longleftarrow \end{matrix} \text{(symm.) monoidal category } \underline{\underline{M}}^\times$

Theorem (E.A., C. LESTER) There is an equivalence of groupoids

$$\text{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n; \mathcal{P}) \simeq \underline{\underline{\text{Pic}}}(\mathcal{V}(\mathcal{T}_1), \dots, \mathcal{V}(\mathcal{T}_n); \mathcal{P})$$

That is, the object $(\mathcal{V}(\mathcal{T}_1), \dots, \mathcal{V}(\mathcal{T}_n))$ of $\underline{\underline{\text{Pic}}}^\times$ corepresents:

$$\text{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n; -) : \underline{\underline{\text{Pic}}}^\times \longrightarrow \text{GRPD}$$

UNIVERSAL MULTIDETERMINANT

Define natural isomorphisms $\det \Rightarrow \det^3 : \mathcal{T}_1 \times \mathcal{T}_2 \times \dots \times \mathcal{T}_n \longrightarrow \mathcal{P}$
 to obtain a groupoid $\text{DET}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n; \mathcal{P})$.

Recall:

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That is, the object $(\mathcal{V}(\mathcal{T}_1), \dots, \mathcal{V}(\mathcal{T}_n))$ of $\underline{\underline{\text{Pic}}}^{\times}$ corepresents:

$$\text{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n; -) : \underline{\underline{\text{Pic}}}^{\times} \longrightarrow \text{GRPD}$$

Theorem (E.A., C. LESTER) For each Picard groupoid \mathcal{P} (*) determines

$$\text{DET}(-; \mathcal{P}) : \underline{\underline{\text{TRCAT}}}_{\text{Verdier}}^{\times \text{ sp}} \longrightarrow \text{GRPD}$$

VERDIER STRUCTURES

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

$$\begin{array}{ccccccc}
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & \Sigma X'' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma X' & \longrightarrow & \Sigma Y' & \longrightarrow & \Sigma Z' & \longrightarrow & \Sigma^2 X'
 \end{array}$$

The η -diagram:

Each line is a distinguished triangle.

VERDIER STRUCTURES

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

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 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & \Sigma X'' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma X' & \longrightarrow & \Sigma Y' & \longrightarrow & \Sigma Z' & \longrightarrow & \Sigma^2 X'
 \end{array}$$

Blue arrows indicate maps: $X' \rightarrow Y$, $Y' \rightarrow Y$, and $Y'' \rightarrow A$.

The η -diagram:

Each line is a distinguished triangle.

Consider the triangle

$$X' \longrightarrow Y \longrightarrow A \longrightarrow \Sigma X'$$

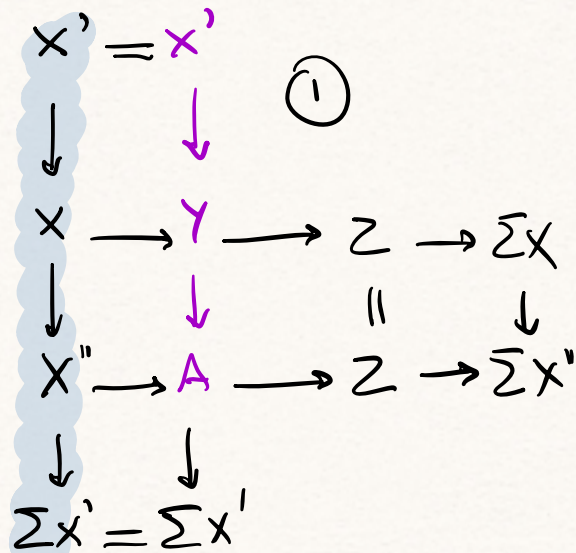
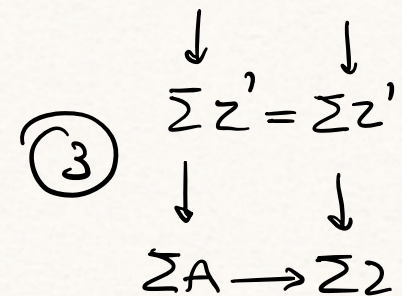
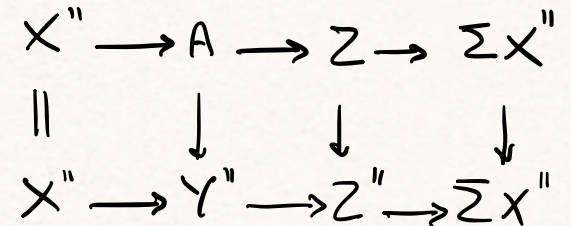
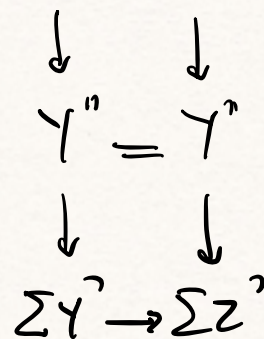
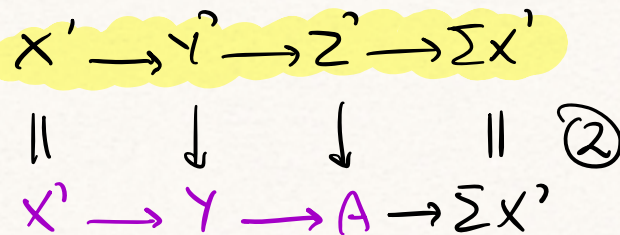
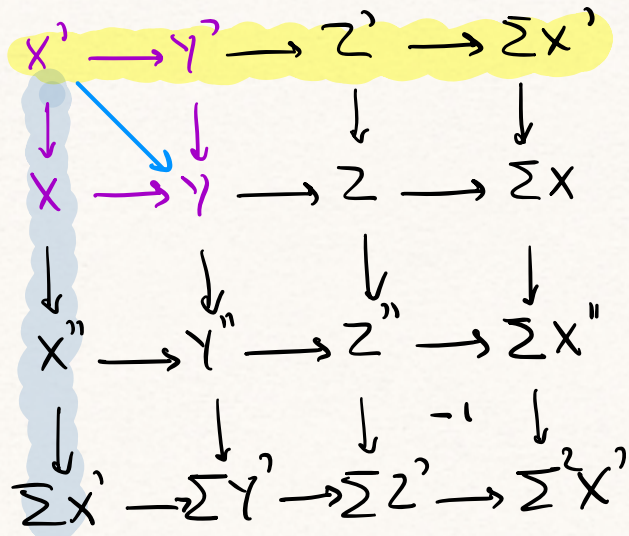
VERDIER STRUCTURES

(Beilinson-Bernstein-Deligne / May / Keller-Neeman)

The 9-diagram has a

Verdier structure

if there exist octahedra



VERDIER STRUCTURES (E.A., C. LESTER)

$F: \mathcal{T}_1 \times \dots \times \mathcal{T}_m \longrightarrow \mathcal{T}$ multiexact admits a Verdier structure

if for all $\Delta_i \in \mathcal{T}_i$, $\Delta_j \in \mathcal{T}_j$, $i < j$, the diagram

$$\begin{array}{ccccccc}
 F(X_i, X_j) & \longrightarrow & F(Y_i, X_j) & \longrightarrow & F(Z_i, X_j) & \longrightarrow & \Sigma F(X_i, X_j) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 F(X_i, Y_j) & \longrightarrow & F(Y_i, Y_j) & \longrightarrow & F(Z_i, Y_j) & \longrightarrow & \Sigma F(X_i, Y_j) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 F(X_i, Z_j) & \longrightarrow & F(Y_i, Z_j) & \longrightarrow & F(Z_i, Z_j) & \longrightarrow & \Sigma F(X_i, Z_j) \\
 \downarrow & & \downarrow & & \downarrow & \quad -1 & \downarrow \\
 \Sigma F(X_i, X_j) & \longrightarrow & \Sigma F(Y_i, X_j) & \longrightarrow & \Sigma F(Z_i, X_j) & \longrightarrow & \Sigma^2 F(X_i, X_j)
 \end{array}$$

admits a Verdier structure

CATEGORICAL RINGS

Def A Picard groupoid $(\mathcal{P}, +, 0)$ is a **categorical ring** if there exists a **second monoidal structure**

$$\cdot : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$$

which is **biexact**, **unital**, **associative**.

This is the **biased** version: associativity, etc. require higher arity structures by composition. Alternatively:

Def A **categorical ring** is a (commutative) monoid in Pic

This is the **unbiased** version.

Remark Similarly, we consider a **TT-cat** $(\mathcal{T}, \otimes, \mathbb{I})$ as a monoid in TRCAT

DET & \otimes

Theorem (E.A., C. Lester)

- $(\mathcal{T}, \otimes, \mathbb{I})$ - Tensor Triangulated cat.
- \otimes admits Verdier

Then the universal Picard groupoid $\mathcal{V}(\mathcal{T})$ is a categorical ring

DET & \otimes

Theorem (E.A., C. Lester)

- $(\mathcal{T}, \otimes, \mathbb{I})$ - Tensor Triangulated cat.
- \otimes admits Verdier

Then the universal Picard groupoid $V(\mathcal{T})$ is a categorical ring

Idea of proof

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \xrightarrow{\otimes} & \mathcal{T} \\ \text{det} \times \text{det} \downarrow & \nearrow & \downarrow \text{det} \\ V(\mathcal{T}) \times V(\mathcal{T}) & \xrightarrow{\otimes} & V(\mathcal{T}) \end{array}$$

$\text{det} \circ \otimes$

\otimes



DET & \otimes

Theorem (E.A., C. Lester)

- $(\mathcal{T}, \otimes, \mathbb{I})$ - Tensor Triangulated cat.
- \otimes admits Verdier

Then the universal Picard groupoid $V(\mathcal{T})$ is a categorical ring

Idea of proof

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \xrightarrow{\otimes} & \mathcal{T} \\ \text{det} \times \text{det} \downarrow & \searrow^{\text{det} \circ \otimes} & \downarrow \text{det} \\ V(\mathcal{T}) \times V(\mathcal{T}) & \xrightarrow{\oplus} & V(\mathcal{T}) \end{array}$$

(A red arrow points from the bottom-left to the top-right, and a red circle is drawn below the bottom arrow.)

As a Corollary, we get the well known fact

$$K_0(\mathcal{T}) \cong \pi_0 V(\mathcal{T}) \quad \text{ring}$$

$$K_1(\mathcal{T}) \cong \pi_1 V(\mathcal{T}) \quad K_0(\mathcal{T})\text{-bimodule}$$

Outlook

What does the Postnikov Invariant

$$\eta_{V(\tau)} \in \pi H^3(\pi_0 V/\tau, \pi_1 V/\tau)$$

Say about (τ, \otimes, I) ?

Outlook

What does the Postnikov Invariant

$$\eta_{V(\tau)} \in \pi_1 \text{HH}^3(\pi_0 V/\mathcal{I}, \pi_1 V/\mathcal{I})$$

Say about $(\mathcal{T}, \otimes, \mathcal{I})$?

THANK YOU!