Groups, Hopf Algebras, Representations & Tannaka-Krein Reconstruction

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Def'n: Let G be a group. A <u>representation</u> of G is a vector space V and a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(V)$$

This is equivalent to a left G-space, i.e. an operation

$$\circ : G \times V \to V$$

s.t.

1) $$\underbrace{g_1 \circ (g_2 \circ v)}_{\text{action}} = (\overbrace{g_1 g_2}^{\text{multiplication in } G}) \cdot v$$

2) $$1_G \cdot v = v$$

Representation Theory is a huge subject, see Fulton Harris - Representation Theory, A First Course.

One key subject: finite-dimensional

A representation is <u>irreducible</u> if it contains no subrepresentations, except 0 and itself.

- Given a group G, find all its irreducible representations.
- If V, W are irreducible representations of G, find the decomposition of $V \otimes W$ into irreducible representations.

Form a category $Rep(G)$

Objects are G-spaces

Morphisms are G-linear maps, i.e. linear maps $f: V \to W$ s.t. furthermore

$$f(g \cdot v) = g \cdot f(v)$$

$Rep_{fd}(G)$ is the full subcategory of f.d. representations.

Thm: $\mathrm{Rep}(G)$ is symmetric monoidal closed

Key constructions:

If V, W are in $|\mathrm{Rep}(G)|$, then $V \otimes W$ is the same underlying vector space with action

$$g\cdot(v \otimes w) = g\cdot v \otimes g\cdot w$$

$V \Rightarrow W$ has the same underlying vector space with $(g\cdot f)(v) = g\cdot f(g^{-1}\cdot v)$

Ex: Verify the details of this. What do you need to do?

The subcategory $\mathrm{Rep}_{fd}(G)$ is also closed, and in fact compact closed.

Compact closed (Kelly-Laplaza)

$$V \Rightarrow W \cong V^{*} \otimes W$$

$$V^{**} \cong V$$

Side Note If M is a monoid, $\mathrm{Rep}(M)$ is still symmetric monoidal closed, but we no longer have that $V \Rightarrow W$ in $\mathrm{Rep}(M)$ has the same underlying vector space as in Vec. It is seen to be closed by the adjoint functor Thm.

The Tannaka-Krein Theorem is about recovering a group from its category of (complex) representations. Original version was with respect to compact (topological) groups or finite groups. We use complex vector spaces.

The key structure is that I have a forgetful (or fibre) functor

$$F : \mathrm{Rep}_{fd}(G) \to \mathrm{Vec}_{fd}$$

I consider all the natural transformations

$$\tau : F \to F$$

which are tensor preserving, i.e.

$$\tau_{V \otimes W} = \tau_V \otimes \tau_W$$

and self conjugate $\bar{\tau} = \tau$

Let $T(G)$ be the set of all such natural transformations.

It has a canonical topology, in the following way.

$\mathrm{End}(F)$ gets a topology by considering $V \in \mathrm{Vec}_{fd}$ $\pi_V : \mathrm{End}(F) \to \underbrace{\mathrm{End}(V)}$

which evaluates a natural transformation at its V-component. $\mathrm{End}(V)$ is a finite dimensional complex vector space

I give $\mathrm{End}(F)$ the coarsest topology making all the π_V's continuous.

Then $T(G) \subseteq \mathrm{End}(F)$ gets the subspace topology.

Thm (Tannaka)

$$G \simeq T(G)$$

This is an isomorphism of compact groups.

Note: We didn't say that the elements of $T(G)$ are invertible. That's a consequence of everything else.

The existence of a fibre functor is crucial in most theorems of this type.

One exception: Doplicher–Roberts Theorem

Thm: Every sufficiently nice monoidal

↑

see nLab

category is equivalent to $Rep_{fd}(G)$ for a unique compact group.

The remarkable thing about this result is the lack of a fibre functor.

The original DR proof is "difficult" to say the least. Michael Muger came up with an alternate proof. He used all the structure of the above theorem to build a fibre functor, reducing the DR theorem to the previous result

The "problem" with groups is that they are not non-commutative enough in the sense that Rep(G) is a symmetric monoidal category.

Physics applications suggested non-symmetric monoidal categories should be studied.

This leads to Hopf algebras

H Hopf algebra is a vector space H with maps

$$\left.\begin{array}{l} m: H\otimes H \to H \\ \eta: k \to H \end{array}\right\} \text{ assoc alg}$$

$$\left.\begin{array}{l} \Delta: H \to H\otimes H \\ \varepsilon: H \to k \end{array}\right\} \text{ co assoc coalg}$$

$$S: H \to H \qquad \text{antipode}$$

satisfying lots of equations

See

Christian Kassel - Quantum Group

<u>EX 1</u> G a group $H = k[G]$

$$H \otimes H \to H$$
$$g_1 \otimes g_2 \mapsto g_1 g_2$$
$$k \to H$$
$$1_k \mapsto 1_G$$
$$H \longrightarrow H \otimes H$$
$$g \mapsto g \otimes g$$
$$H \to k$$
$$g \mapsto 1_k \qquad S(g) = g^{-1}$$

<u>EX 2</u> G a <u>finite</u> group

$$H \otimes H \to H$$
$$g_1 \otimes g_2 \mapsto \begin{cases} g_1 & \text{if } g_1 = g_2 \\ 0 & \text{if not} \end{cases}$$
$$k \to H$$
$$1_k \mapsto \sum_{g \in G} g$$
$$H \to H \otimes H$$
$$g \mapsto \sum_{g_1 g_2 = g} g_1 \otimes g_2$$

$$H \to k$$

$$g \mapsto \begin{cases} 1_k & \text{if } g = 1_G \\ 0 & \text{if not} \end{cases}$$

$$S(g) = g^{-1}$$

EX3 Let X be a set thought of as an alphabet. Let X^* be the free monoid generated by X. Call the elements words

If $w_1, w_2 \in X^*$ a shuffle is a permutation of words w_1 and w_2 such that the internal order of w_1 and w_2 are maintained

Ex $w_1 = ab$ $w_2 = cd$

Shuffles are abcd

acbd

acdb

cabd

cadb

cdab

Let $H = k[X^*]$

$m : H \otimes H \to H$

$$m(w_1 \otimes w_2) = \sum_{w \mid w \text{ is a shuffle}} w$$

$\eta : k \to H$

$1_k \mapsto \epsilon$ (the empty word

$\Delta : H \to H \otimes H$

$$w \mapsto \sum_{w = w_1 w_2} w_1 \otimes w_2$$

concatenation

<u>EX</u>: $abc \mapsto \epsilon \otimes abc + a \otimes bc + ab \otimes c + abc \otimes \epsilon$

$\varepsilon : H \to k$

$$\varepsilon(w) = \begin{cases} 0 & \text{if } w \neq \epsilon \\ 1 & \text{if } w = \epsilon \end{cases}$$

$S(w) = (-1)^{|w|} \overline{w}$

w backwards

EX1 check the details of this, i.e. all equations

EX2 There is a dual Hopf algebra, where the shuffle is used to define comultiplication. Work out details.

If H is a Hopf algebra, an H-module is a vector space M and a map

$$\rho : H \otimes M \to M$$

s.t.

$$\begin{tikzcd} H \otimes H \otimes M \arrow[r, "id \otimes \rho"] \arrow[d, "m \otimes id"'] & H \otimes M \arrow[d, "\rho"] \\ H \otimes M \arrow[r, "\rho"'] & M \end{tikzcd}$$

$$\begin{tikzcd} k \otimes M \arrow[r, "\eta \otimes id"] \arrow[dr, "\sim"', equal] & H \otimes M \arrow[d, "\rho"] \\ & M \end{tikzcd}$$

Thm: If H is a Hopf algebra, $\mathrm{Rep}(H)$ is monoidal.

If M, N are in $\mathrm{Rep}(H)$, so is $M \otimes N$ with action given by

$$H \otimes M \otimes N \xrightarrow{\Delta \otimes \mathrm{id}} H \otimes H \otimes M \otimes N \xrightarrow{c_{23}} H \otimes M \otimes H \otimes N$$
$$\xrightarrow{\rho \otimes \rho} M \otimes N \quad \square$$

H is <u>cocommutative</u> if

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & H \otimes H \arrow[d, "c"] \\ & H \otimes H \end{tikzcd}$$

Thm If H is <u>cocommutative</u>, $\mathrm{Rep}(H)$ is symmetric monoidal.

Converse is false but in general $\mathrm{Rep}(H)$ is not symmetric

Can the Tannaka-Krein theorem be extended to this setting?

Yes. See K.H. Ulbrich -
On Hopf Algebras & Rigid Monoidal Categories
Result goes back to work of P. Deligne
Again, we assume a fibre functor

$$\omega : \mathcal{C} \to Vec_{fd}$$

Ulbrich calls this $\mathcal{P}(k)$

We'll use a series of lemmas due to Deligne. We have a functor

$$Vec \to Ens$$

$$M \mapsto Nat(\omega, \omega \otimes M)$$

Lemma: It's representable.

So $\exists$ a vector space A s.t.

$$\rho : \mathrm{Hom}(A, M) \cong \mathrm{Hom}(w, w \otimes M)$$

This is a difficult result. We'll use ρ repeatedly.

Let $M = A$

$$\rho_A : \mathrm{Hom}(A, A) \cong \mathrm{Hom}(w, w \otimes A)$$

Let $\alpha = \rho_A(\mathrm{id}) : w \to w \otimes A$

Consider

$$w \xrightarrow{\alpha} w \otimes A \xrightarrow{\alpha \otimes \mathrm{id}} w \otimes A \otimes A$$

Applying ρ^{-1} to this gives

$$A \to A \otimes A$$

Consider $e : w \to w \otimes k$ the iso.

Under ρ, we get a map

$$A \to k$$

This gives the coalgebra structure

Lemma: The functor

$$\mathrm{Hom}(w \otimes w, w \otimes w \otimes M)$$

is representable by $A \otimes A$, so

$$p^2 : \mathrm{Hom}(A \otimes A, M) \cong \mathrm{Hom}(\omega \otimes \omega, \omega \otimes \omega \otimes M)$$

Consider

$$\omega(X) \otimes \omega(Y) \cong \omega(X \otimes Y) \xrightarrow{\alpha} \omega(X \otimes Y) \otimes A$$
$$\cong \omega(X) \otimes \omega(Y) \otimes A$$

Letting $M = A$ in the above, gives

$$A \otimes A \longrightarrow A$$

unit is similar.

To obtain antipode, use compact closed structure.

<u>Note</u>: All of these results are based specifically in categories of vector spaces.

But such results for more abstract categories exist.