Many-Valued Topology

References:

U. Höhle - Many-valued topology and its applications

U. Höhle - Categorical foundations of topology with application to quantaloid enriched topological spaces.

If X is a set, a filter on X is a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ s.t.

- $X \in \mathcal{B}$
- $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$
- $A \in \mathcal{B}, A \subseteq B \Rightarrow B \in \mathcal{B}$

$\mathbb{F}(X)$ = set of all filters in X

Thm: It's a monad

$$\eta: X \to \mathbb{F}(X)$$

$$x \mapsto \{A \subseteq X \mid x \in A\} \quad \text{principal filter}$$

$$\mu: \mathbb{F}^2(X) \to \mathbb{F}(X)$$

is more complicated

Defn: A <u>nhood system on a set X</u> is an an arrow in the Kleisli category $U : X \to X$

So $U : X \to \mathbb{F}X$

such that

- $U(x) \subseteq \eta_X(x)$
- $U \circ U = U$

Kleisli composition

Defn: A <u>topological space</u> is a pair $(X, \mathcal{U})$ with $\mathcal{U}$ a nhood system

Lem: This is equivalent to usual definition.

This leads to <u>monadic topology</u>

But note we also need an ordering on hom-sets in the Kleisli category.

We need the base category to be sufficiently well-behaved.

Factorization systems

In Set, given any map $f: X \to Y$ it can be factored

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{epi}"'] & & Y \\ & \mathrm{Im}(f) \arrow[ur, "\text{monic}"'] & \end{tikzcd}$$

So in general, we'll have 2 classes of maps E and M. They are frequently epis & monics, but not always. So some books use L and R instead.

Properties

1) Both E and M contain all isos and are closed under composition

2) Every morphism can be factored as $f = m \circ e$ where $e \in E$ and $m \in M$.

3) Suppose I have the following

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[r, "m"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"] & \cdot \arrow[r, "m'"] & \cdot \end{tikzcd}$$

with $e, e' \in E$, $m, m' \in M$, u, v arbitrary then is a unique fill-in in the middle

<u>Alternate def'n</u>

Suppose I have the following set up

$$\begin{tikzcd} \cdot \arrow[r, "e \in E"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \arrow[dl, dashed] \\ \cdot \arrow[r, "m \in M"'] & \cdot \end{tikzcd}$$

If there is a unique diagonal making both triangles commute, we'll write $e \downarrow m$. If $H \subseteq \mathrm{Ar}(\mathcal{C})$, write

$$H^{\uparrow} = \{ e \mid \forall h \in H, e \downarrow h \}$$

$$H^{\downarrow} = \{ m \mid \forall h \in H, h \downarrow m \}$$

The 3rd condition is equiv to

$$E \subseteq M^{\uparrow} \quad \text{and} \quad M \subseteq E^{\downarrow}$$

A monic q is extremal if in every epi-mono factorization

$$q = m \circ e.$$

the epi must be an iso

~~When~~ What we need:

Our category $\mathcal{C}$ must have the property that

$$(\mathcal{E}, \mathcal{M})$$

epis ↑ ↑ extremal monics

forms a factorization system

In particular, we need an epi-monic factorization system. See the references.

This structure allows you to define partial orders on an object

$$\leq \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \quad \text{or} \quad \leq \rightarrowtail A \times A$$

There is sufficient structure to define reflexive, transitive and antisymmetric.

Eg: Antisymmetry says:

Consider the pullback of:

$$\begin{tikzcd}
 & \leq \arrow[d, "\langle \beta, \alpha \rangle"] \\
\leq \arrow[r, "\langle \alpha, \beta \rangle"'] & A \times A
\end{tikzcd}$$

This should correspond to

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[d, "\sigma"'] \arrow[dr, dashed, "\Delta"'] & \leq \arrow[d, "\langle \beta, \alpha \rangle"] \\
\leq \arrow[r, "\langle \alpha, \beta \rangle"'] & A \times A
\end{tikzcd}$$

Similarly, one can talk about monotone maps

$$\varphi : (A, \leq_A) \longrightarrow (B, \leq_B)$$

$$\begin{tikzcd}
\leq_A \arrow[r, "\exists \rho"] \arrow[d] & \leq_B \arrow[d] \\
A \times A \arrow[r, "\varphi \times \varphi"'] & B \times B
\end{tikzcd}$$

So we have a category $Po(\mathcal{C})$

<u>Thm:</u> A pair of arrows $\leq \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A$ is a partial order on A iff $\langle \alpha, \beta \rangle \longrightarrow A \times A \in \mathcal{M}$

and for every object X, we can define a partial order on $Hom(X, A)$ by

$$\varphi \leq \psi \iff \exists\, X \xrightarrow{\ell} \leq \text{ s.t.}$$

$$\varphi = \alpha \circ \ell \quad \text{and} \quad \psi = \beta \circ \ell$$

See the book by Höhle

So in fact

Thm: $Po(\mathcal{C})$ is a ~~2-category~~ locally posetal bicategory

We have a forgetful functor

$$V : Po(\mathcal{C}) \to \mathcal{C}$$

We'll say an endofunctor $T : \mathcal{C} \to \mathcal{C}$ <u>factors through</u> <u>V</u> if $\exists S : \mathcal{C} \to Po(\mathcal{C})$ s.t. $T = V \circ S$

Def An <u>ordered monad</u> is a monad (T, η, μ) s.t.

1) T factors through V as above
2) Kleisli composition is monotone in each variable
3) $\forall$ objects X μ_X has a left adjoint in $\mathcal{C}_T$

The last statement uses the fact that $\mathcal{C}_T$ is a locally posetal bicategory

<u>EX</u>: Note the following terminology is terrible, and not my fault.

A complete groupoid is a complete lattice L and a binary operation $\otimes : L \times L \to L$ which is monotone in each variable (no assumption of associativity, unit or $\otimes$ distributing over sups or infs)

We'll denote the top element by 1 and bottom by 0.

Similarly if L is a complete groupoid, so is L^X where X is a set, with pointwise order & multiplication. We'll denote the top & bottom elements of this groupoid by 1_X and 0_X

Def'n: An L-valued filter is a function $r : L^X \to L$ s.t.

1) $r(1_X) = 1$ (top for L)

2) $f_1 \leq f_2 \Rightarrow r(f_1) \leq r(f_2)$

3) $r(f_1) \otimes r(f_2) \leq r(f_1 \otimes f_2)$

4) $r(0_X) = 0$ (bottom for L

Let $F_L(X)$ = set of all L-filters on X

Thm: ~~B~~ F_L is a partially ordered monad.

<u>Aside</u> To verify that we have a monad, there are 3 methods.

1) Just verify the original equations, as in MacLane

But this is difficult in general. In particular,

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

is difficult.

2) Kleisli Triple (due to Ernie Manes)

- A function $T : ob(\mathcal{C}) \to ob(\mathcal{C})$
- $\forall A \in ob(\mathcal{C})$, $\eta : A \to TA$
- $$\frac{A \xrightarrow{f} TB}{TA \xrightarrow{f^{\#}} TB}$$

satisfying various equations

3- A <u>monad in clone form</u>

$$\left.\begin{array}{l} \cdot\ T : ob(\mathcal{C}) \longrightarrow ob(\mathcal{C}) \\ -\ \eta_x : X \to TX \\ -\ \text{clone composition} \\ \dfrac{X \xrightarrow{f} TY \qquad Y \xrightarrow{g} TZ}{g \circ f : X \to TZ} \end{array}\right\} \text{various equations}$$

<u>EX 2</u> Let Q be a quantale.

$Q^{(-)}$ is a monad.

<u>Method 3</u> $\dfrac{X \longrightarrow Q^Y}{X \times Y \to Q}$

So clone composition is the usual Q-Rel composition

~~But~~ $\eta : X \to Q^X$ is

$$\eta(x) = \delta_x : y \longmapsto \begin{cases} e & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

e is the unit

On arrows

$$\frac{f: X \to Y}{Q^f : Q^X \to Q^Y}$$

If $h: X \to Q$ and $y \in Y$

$$Q(f)(h)(y) = \bigvee_{x \in X \mid f(x) = y} h(x)$$

To define $\mu_X : Q^{Q^X} \longrightarrow Q^X$

If ~~x~~ $x \in X$, $g: Q^X \to Q$, define

$$\mu(g)(x) = g(\delta_x)$$

Now we can talk about topological spaces in general categories. (Chapter 3 of Höhle.)

Defn: Suppose $\mathcal{C}$ is a category

We assume $\mathcal{C}$ is equipped with a partially ordered monad.

Let X be an object of $\mathcal{C}$, and $v: X \to TX$ an endomorphism in the Kleisli category of ~~T~~ T

The pair (X, v) is a <u>semitopological space</u> if

(T1): $v \leq \eta_X$

Note this is with respect to the induced ordering in $\mathrm{Hom}_{\mathcal{C}}(X, TX)$

(X, v) is a <u>topological space</u> if, furthermore

$$v \leq v \circ v$$

↑ Kleisli composition

Note this implies $v = v \circ v$

Now we ask what amount of topology can we do in this abstract setting.

First up: Continuous maps

Let (X, v_1) and (Y, v_2) be topological spaces in $\mathcal{C}$.

A map $f: X \to Y$ in $\mathcal{C}$ is <u>continuous</u>

$$v_2 \circ (\eta_Y \cdot \varphi) \leq (\eta_Y \cdot \varphi) \circ v_1$$

Note $\circ$ is Kleisli composition

$\cdot$ is $\mathcal{C}$ composition

$$\frac{X \xrightarrow{\varphi} Y \xrightarrow{\eta_Y} TY \qquad Y \xrightarrow{v_2} TY}{X \to TY}$$

$$\leq$$

$$\frac{X \xrightarrow{v_1} TX \qquad X \xrightarrow{\varphi} Y \xrightarrow{\eta_Y} TY}{X \to TY}$$

Result is a category we'll call $\mathcal{C}_T$-Top

One can define

- dense subspace
- closure
- hausdorff
- regular
- compactness