Logic Seminar

Main Reference

Applications of Sheaves
Springer Lecture Notes in Mathematics
#753, available for download from
our library.

Especially the article
Sheaves & Logic by Fourman & Scott

Def'n: A <u>sheaf</u> is something you can attach to a topological space which keeps track of local data. Sheaves can be used to analyze properties of that space

<u>Ex:</u> Let X be a space and $\mathcal{O}(X)$ the poset of open sets, ordered by inclusion

Consider the assignment:

$$U \in \mathcal{O}(X) \mapsto F(U) \quad \text{where}$$

$$F(U) = \{ f : U \to \mathbb{R} \mid f \text{ is continuous} \}$$

Note If X is a manifold, you could take the smooth maps, etc.

If $U \subseteq V$, we have a <u>restriction map</u>

$$F_{U,V} : F(V) \to F(U)$$

which restricts a continuous map $f : V \to \mathbb{R}$ to the subspace of the domain U.

The restriction maps are <u>functorial</u>

$$U \subseteq V \subseteq W$$

$$\begin{tikzcd}
F(W) \arrow[r] \arrow[dr] & F(V) \arrow[d] \\
 & F(U)
\end{tikzcd}$$

A <u>presheaf</u> is a contravariant functor

$$F : \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$$

poset viewed as a category.

could also be Ring or other choices

Key properties

Let $U \subseteq X$ be open and $\{U_i\}_{i \in I}$ an open cover of U,

i.e. $U = \bigcup_{i \in I} U_i$

Suppose $\forall i \in I$, I have $f_i : U_i \to \mathbb{R}$, i.e. $f_i \in F(U_i)$. Suppose this family of maps "agree on overlaps" (formalize this yourselves), then there is a <u>unique</u> $f : U \to \mathbb{R}$ s.t.

$$F_{U,U_i}(f) = f_i$$

Abstracting this property gives the definition of sheaf.

Thm: Both $PreSh(X)$ and $Sh(X)$ are cartesian closed categories

$$Sh(X) \subseteq PreSh(X)$$

is a reflective subcategory.

In fact, both are toposes.

<u>Locales</u>: "Pointless topology". Replace $\mathcal{O}(X)$ with more general posets*.

Def'n: A <u>locale</u> is a poset with all suprema (least upper bounds) [and hence all infima (greatest lower bounds)] s.t.

$$a \wedge \left(\bigvee_{i \in I} U_i \right) = \bigvee_{i \in I} (a \wedge U_i)$$

"finite infs commute with arbitrary sups"

See Peter Johnstone's book <u>Stone Spaces</u>

If $f: X \to Y$ is a continuous map, then $f^{-1}: \mathcal{O}(Y) \to \mathcal{O}(X)$ is a map of locales.

Def'n: Define the category Loc to have locales as objects. A morphism $p: L \to M$ in Loc is a function $p: M \to L$ preserving finite infs and all sups.

Lemma: ~~[illegible]~~ $\mathcal{O}: \mathrm{Top} \to \mathrm{Loc}$ is a functor.

Other direction:

Let $\mathbb{2} = \{0 < 1\}$ or $\{F < T\}$ and let $\mathcal{U}$ be a locale. A point of $\mathcal{U}$ is an arrow in Loc

~~$x: \mathbb{2} \to \mathcal{U}$~~ $x: \mathcal{U} \to \mathbb{2}$

~~[illegible]~~ which preserves locale structure.

Let $pt(\mathcal{U})$ be the points of $\mathcal{U}$

A point x is determined by the set

$$\mathcal{U}_x : \{ U \in \mathcal{U} \mid x(U) = 1 \}$$

If $U \in \mathcal{U}$, let $\tilde{U} \subseteq Pt(\mathcal{U})$ be the set of all points $x \in Pt(\mathcal{U})$ s.t. $U \in \mathcal{U}_x$.

So $\tilde{U} \subseteq Pt(\mathcal{U})$

Lemma: The sets $\tilde{U}$ for $U \in \mathcal{U}$ is a basis for a topology in $Pt(\mathcal{U})$

This construction determines a functor

$Pt : Loc \to Top$ which

is ~~left~~ right adjoint to

$\mathcal{O} : Top \to Loc$

But what does this have to do with logic?

Lemma: Let L be a locale. The operation $a \wedge (-)$ has a right adjoint for all $a \in L$. Denote it $a \Rightarrow$.

So it satisfies $a \wedge b \leq c$ iff $b \leq a \Rightarrow c$

Proof: Follows from the general adjoint functor thm, or just write down the formula.

Fact: A locale is a Heyting algebra (in fact a complete Heyting algebra) and Heyting algebras are the poset al models of intuitionistic logic.

classical logic $\Leftrightarrow$ Boolean algebra

intuitionistic logic $\Leftrightarrow$ Heyting algebra

We want to generalize sheaves on a space to sheaves on a locale. There is an equivalent formulation that makes use of the logical structure of locales.

We're now in Fourman & Scott paper, p 349

Back to our main example. If $a : U \to \mathbb{R}$ is continuous with $U \subseteq X$, we'll write $Ea = U$. This is called the <u>extent</u> of a. It measures the "time" for which a exists. The function a is to be thought of as a <u>variable quantity</u>.

Given 2 variable quantities

$$a : U \to \mathbb{R}$$
$$b : V \to \mathbb{R}$$

we define the extent to which they are equal by

$$[\![a = b]\!] = \mathrm{Int}\{ t \in Ea \cap Eb \mid a(t) = b(t) \}$$

So if we are thinking of $\mathcal{O}(X)$ as our truth value algebra, then this is the truth value of the statement $a = b$.

$[\![\cdot = \cdot]\!]$ is called the <u>equality map</u>.

Note in particular that $Ea = [\![a = a]\!]$

Defn: Let Ω be a locale. An Ω-Set is a set $|A|$ equipped with an <u>Ω-valued equality map</u> $[\![\cdot = \cdot]\!] : |A| \times |A| \to \Omega$ s.t.

1) $[\![a = b]\!] = [\![b = a]\!]$

2) $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

Given an Ω-Set, we define

$$Ea = [\![a = a]\!]$$

Def'n: Let Ω be a locale. An <u>Ω-presheaf</u> is a set $|A|$ with maps

$$E : |A| \to \Omega \quad \text{extent}$$

$$-|_{-} : |A| \times \Omega \to A \quad \text{restriction}$$

s.t.

1) $a|_{Ea} = a$

2) $(a|_p)|_q = a|_{p \wedge q}$

3) $E(a|_p) = Ea \wedge p$

Lemma: Every Ω-presheaf is an Ω-set where

$$[\![a = b]\!] = \bigvee \{ p \leq Ea \wedge Eb \mid a|_p = b|_p \}$$

But note, not every Ω-set determines an Ω-presheaf, since there won't be an obvious restriction.

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Defn: Suppose A is a presheaf and $a, b \in |A|$. 1) We'll say a is a restriction of b and write $a \leq b$ if

$$a = b|_{Ea}$$

2) If $B \subseteq |A|$ is any subset, we say B is compatible iff the elements are pairwise compatible in the sense that

$$b|_{Eb'} = b'|_{Eb}$$

Lemma: 1) $\leq$ defines a partial order on $|A|$

2) $b \leq a \Rightarrow Eb \leq Ea$

Defn: 1) If B is a compatible family, a join for B is a minimal upper bound for B under $\leq$.

2) A presheaf is separated

if joins of subsets, when they exist, are unique.

Def'n: Let A be an Ω-set

Define $[\![a \equiv b]\!] = (Ea \vee Eb) \Rightarrow [\![a = b]\!]$

An Ω-set is <u>separated</u> iff

$[\![a \equiv b]\!] = T \Rightarrow a = b$

Prop: A, a presheaf, is separated iff A, viewed as an Ω-set, is separated.

See p 342 for a # of equivalences.

Thm: TFAE

1) A is separated

2) A is separated as an Ω-set

3) $a|_{[\![a=b]\!]} = b|_{[\![a=b]\!]} \quad \forall a,b \in |A|$

4) $a|_{[\![a\equiv b]\!]} = b|_{[\![a\equiv b]\!]} \quad \forall a,b \in |A|$

EX: Let Ω be a locale. Ω is an Ω-set if ~~s~~ we defined

$$[\![p = q]\!] = p \leftrightarrow q = (p \to q) \wedge (q \to p)$$

This is a separated Ω-set and not a presheaf.

Def'n: A presheaf is a sheaf if every compatible family ~~s~~ has a unique join.

Lemma: Every sheaf is separated.

When does an Ω-set correspond to a sheaf?

Def'n: Let A be an Ω-set. A singleton of A is a map $s: |A| \to \Omega$ such that

1) $s(a) \wedge [\![a=b]\!] \leq s(b)$

2) $s(a) \wedge s(b) \leq [\![a=b]\!]$

Ex: Let $c \in |A|$. It determines a map

$$\tilde{c} = [\![\cdot = c]\!] : |A| \to \Omega$$

Def'n: An Ω-set is complete if every singleton determines a unique element, i.e. if $s : |A| \to \Omega$ is a singleton, $\exists!$ $c \in |A|$ such that $s = \tilde{c}$.

Thm: Sheaves and complete Ω sets are the same thing.

Functoriality? For any presheaf A over Ω and for $p \in \Omega$, define

$$A(p) = \{a \in |A| \mid E(a) = p\}$$

Then restriction gives us maps

$$a \mapsto a|_q$$

$$\rho^p_q : A(p) \to A(q)$$

It's easy to check if $r \le q \le p$, then

$$\begin{tikzcd} A(p) \arrow[r] \arrow[dr] & A(q) \arrow[d] \\ & A(r) \end{tikzcd}$$

So we have a functor $A : \Omega^{op} \to Set$

Conversely, given such a functor, we can build a presheaf (Exercise)

<u>Next Up</u>

- When locales are replaced by quantales the above nice correspondence fails
- If we replace sets by finiteness spaces, we can use rings rather than quantale