$$
\begin{tikzcd}
 & (x, g\circ f_1) \arrow[r, mapsto] & g(f_1(x)) \arrow[dr, mapsto] & \\
(x,g) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \arrow[d, phantom, "\checkmark"] & f_2(g(f_1(x))) \\
 & (f_1(x), f_2\circ g) \arrow[r, mapsto] & (f_2\circ g\circ f_1)(x) \arrow[ur, mapsto] &
\end{tikzcd}
$$

Same proof works in Vec. A more diagrammatic proof works in any closed category

<u>Exercise</u> In any closed category, we also have a co-evaluation

$$X \to Y \Rightarrow (X \times Y)$$

This fails to be natural for the same reason

$$x \mapsto [y \mapsto (x,y)]$$

Carry out the same reasoning to show it is dinatural.

Let $\mathcal{C}$ be any category.

Consider $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$

Let n be a natural number. We have a dinatural transformation

$$\tau_n : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$$

given by

$$\tau_n(f) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

Verify this!

It was conjectured that when $\mathcal{C} = \mathrm{FinSet}$, all dinaturals were of this form.

This was proven false by Paré & Román.

See their paper "Dinatural Numbers"

Lemma: Every natural transformation is dinatural.

given $F, G : \mathcal{C} \to \mathcal{D}$

define $\hat{F}, \hat{G} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$

$$\hat{F}(X, X') = F(X)$$
$$\hat{G}(X, X') = G(X)$$

The hexagon becomes just the usual square.

- Suppose F is a constant functor. Then the left hand side of the hexagon degenerates into a constant. I get

$$\begin{tikzcd} & GXX \arrow[dr, "GXf"] & \\ C \arrow[ur, "\theta_X"] \arrow[dr, "\theta_Y"'] & & GXY \\ & GYY \arrow[ur, "GfY"'] & \end{tikzcd} \qquad f : X \to Y$$

where I have a family of maps

$$\theta_X : C \to GXX$$

This is called a <u>wedge</u>

EX in Sets

Let $F(X,Y) = \{*\}$ $\quad G(X,Y) = X \Rightarrow Y$

$\theta_X : F(X,X) \to G(X,X)$

$* \mapsto id_X$

$$\begin{tikzcd}
 & X \Rightarrow X \arrow[dr, "f\circ -"] \arrow[dd, phantom, "\checkmark"] & \\
* \arrow[ur] \arrow[dr] & & X \Rightarrow Y \\
 & Y \Rightarrow Y \arrow[ur, "-\circ f"'] &
\end{tikzcd}$$

Both legs give $f : X \to Y$

If G is a constant functor, I get a

<u>co-wedge</u>

$$\begin{tikzcd}
 & F(X,X) \arrow[dr] & \\
F(Y,X) \arrow[ur] \arrow[dr] & & C \\
 & F(Y,Y) \arrow[ur] &
\end{tikzcd}$$

Ex in Vec

$$F(V,W) = V^* \otimes W$$

$$G(V,W) = k \quad \text{(base field)}$$

$$\theta_V : V^* \otimes V \to k$$

$$f \otimes v \mapsto f(v)$$

Easy to check cowedge diagram (Do it.)

Lots of examples in the world.

So, here's an idea! Form the following "category": For fixed $\mathcal{C}, \mathcal{D}, n \in \mathbb{N}$, we'll have a category whose objects are n-ary multivariant functors.

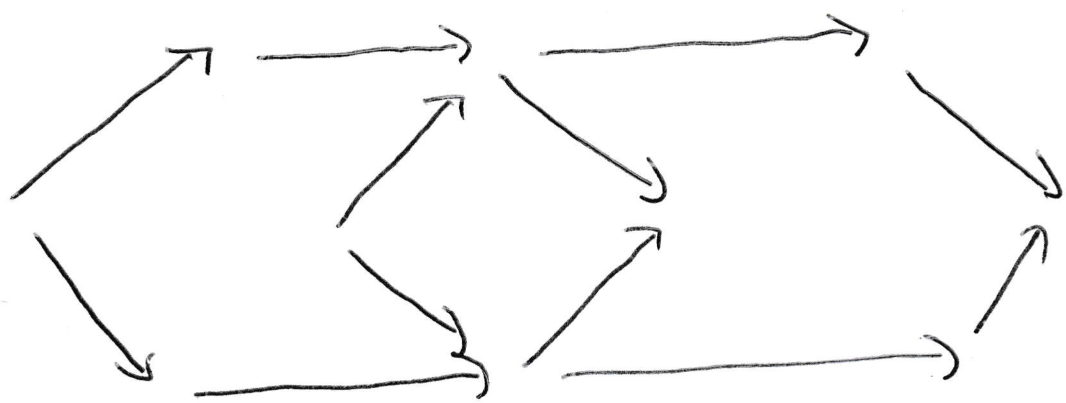
$$F : (\mathcal{C}^{op})^n \times \mathcal{C}^n \to \mathcal{D}$$

An arrow F, G will be a dinatural transformation.

This is an excellent idea, except

IT DOESN'T WORK!

In general dinatural transformations do not compose



It's entirely possible for the two inner hexagons to compose but not the outer.

2 examples

1) In FinSet, the category of finite sets and functions, I have a wedge

$$\theta : \{*\} \longrightarrow F(X,Y) = X \Rightarrow Y$$

I have a co-wedge

$$\underbrace{F(X,Y) = X \Rightarrow Y \longrightarrow \text{Bool} = \{T, F\}}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{constant functor}$$

$$[f : X \to X] \longmapsto \begin{cases} T & \text{if } \# \text{ of fixed points of } f \text{ is even.} \\ F & \text{if } \# \text{ of fixed points of } f \text{ is odd} \end{cases}$$

If you've never seen this before, you probably don't believe it's a dinat. Prove that it is.

$$\{*\} \longrightarrow F(X,Y) \longrightarrow \text{Bool}$$

The composite would be a dinat between constant functors.

which should be a single map.

$$\{x\} \longmapsto \mathrm{id}_X \longmapsto \begin{cases} T & \text{if } |X| \text{ is even} \\ F & \text{if } |X| \text{ is odd} \end{cases}$$

so the composite is not a dinat.

<u>EX 2</u> In Vec_{fd}

$$\theta : k \longrightarrow F(V,W) = V \multimap W$$

$$1 \mapsto \mathrm{id}_V$$

But $V \multimap W \cong V^* \otimes W$

This yields a dinat

$$1 \mapsto \sum_{e_i \in \mathrm{Basis}(V)} e_i^* \otimes e_i$$

You need to check this works, i.e. is still dinatural.

$$\Psi : F(V,W) = V^* \otimes W \to k$$

$$\mathrm{ev} : V^* \otimes V \to k$$

The composite should be a dinat between constant functors.

But I get

$$k \to V^* \otimes V \xrightarrow{?} k$$

What is this composite?

So, you can't always compose dinats.

But sometimes you can.

Referenes

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