Dinaturality, Lecture 1

Prerequisites

Basic Category Theory

- Categories
- Functors
- Natural Transformations
- Adjunctions
- Closed Categories
  - cartesian closed
  - monoidal closed

We'll start from scratch with sequent calculus

Convolution, simple case

Let M be a monoid. Consider

$$\mathrm{Func}(M, \mathbb{C}) = \{ f : M \to \mathbb{C} \}$$

If $f, g \in \mathrm{Func}(M, \mathbb{C})$, define

$f * g : M \to \mathbb{C}$ the convolution product by

$(f * g)(m)$

$$= \sum_{(m_1, m_2) \mid m_1 m_2 = m} f(m_1) g(m_2)$$

Notes

1) This may not be well-defined. The sum could be divergent.

2) There are assumptions one can add to ensure convergence. The most obvious being that M be finite.

3) When defined, it is associative.

$$[(f * g) * h](m) = [f * (g * h)](m)$$

$$= \sum_{(m_1, m_2, m_3) \mid m_1 m_2 m_3 = m} f(m_1) g(m_2) h(m_3)$$

4) It also has a unit. Ex: Figure it out.

5) Both the multiplication in M and the multiplication in $\mathbb{C}$ are used

6) In more complicated (topological) settings, the sum above becomes an integral.

Day Convolution

Suppose $\mathcal{M}$ and $\mathcal{C}$ are monoidal categories. Let Func$(\mathcal{M}, \mathcal{C})$ be the category of all functors between them. Day convolution assigns a monoidal structure to this category. The sum becomes a co-end which is denoted with an integral symbol.

$$\int^{c \in |\mathcal{C}|}$$

This is not a coincidence. Coends behave a lot like integrals, e.g. there is a Fubini Theorem

Understanding Day Convolutions requires understanding

- dinaturality
  - composition of dinaturals and sequent proofs as dinaturals
- ends & co-ends
- Kan Extensions
- What else??

Dinaturality

In the category of Sets & functions, SET, we have an evaluation operation

$$X \times (X \Rightarrow Y) \xrightarrow{ev} Y$$
$$(x, f: X \to Y) \longmapsto f(x)$$

This is the counit of the adjunction making Set a cartesian closed category

There's a similar operation in the category of vector spaces & linear maps

$$V \otimes (V \multimap W) \xrightarrow{ev} W$$

or in any (cartesian or monoidal) closed category.

Is it a natural transformation (in Set)?

Intuition: Yes. (NTs are those operations which have a canonical flavor.)

Answer: NO, for a pretty ridiculous reason. Consider the <u>operation</u>

$$\mathcal{C}^2 \to \mathcal{C}$$

$$(X,Y) \mapsto X \times (X \Rightarrow Y)$$

The variable X appears both covariantly & contravariantly.

So there is no way to define this operation on arrows.

Fix Y and suppose I have a map $f: X \to X'$. Can I build a map?

covariant $X \times (X \Rightarrow Y) \longrightarrow X' \times (X' \Rightarrow Y)$

contravariant

$$X' \times (X' \Rightarrow Y) \longrightarrow X \times (X \Rightarrow Y)$$

Just try it in Set.

But there is a sense in which it is natural.

Defn: Suppose I have functors

$$F, G: (\mathcal{C}^{op})^n \times \mathcal{C}^n \longrightarrow \mathcal{D}$$

I separate out the covariant and contravariant occurences of each variable

A dinatural transformation from

F to G

consists of a family of maps

$$\theta_{\bar{X}} : \underbrace{F\bar{X}\bar{X} \longrightarrow G\bar{X}\bar{X}}$$

I've equated the covariant
and contravariant occurences

indexed by

$\bar{X} = \langle X_1, X_2, \ldots, X_n \rangle$ n-tuples of

objects in $\mathcal{C}$, such that, given
an n-tuple of arrows in $\mathcal{C}^n$

$\bar{f} = \langle f_1, f_2, \;\; , f_n \rangle$

$\bar{f} : \bar{X} \to \bar{Y}$

the following hexagon commutes:

I'll drop the bars

$$\begin{tikzcd}
 & FXX \arrow[r, "\theta_X"] & GXX \arrow[dr, "GXf"] & \\
FYX \arrow[ur, "FfX"] \arrow[dr, "FYf"'] & & & GXY \\
 & FYY \arrow[r, "\theta_Y"'] & GYY \arrow[ur, "GfY"'] &
\end{tikzcd}$$

Thm: Evaluation is a dinatural transformation in any cartesian closed category.

We'll just prove it in Set.

The functors are $F(X_1, X_2, Y_1, Y_2) = Y_1 \times (X_1 \Rightarrow Y_2)$

$$G(X_1, X_2, Y_1, Y_2) = Y_2$$

Note I don't use every variable in either formula. That's OK (and typical).

I get

$$FXX = X_1 \times (X_1 \Rightarrow X_2)$$

$$GXX = X_2$$

So $\theta_X : X_1 \times (X_1 \Rightarrow X_2) \to X_2$

is evaluation

We'll prove the hexagon commutes. Let's calculate the other 4 arrows.

$\nearrow$ $FfX : FYX \to FXX$

$$X_1 \times (Y_1 \Rightarrow X_2) \to X_1 \times (X_1 \Rightarrow X_2)$$

$$(x, g : Y_1 \to X_2) \mapsto (x, g \circ f_1 : X_1 \to X_2)$$

$\searrow$ $FYf : FYX \to FYY$

$$X_1 \times (Y_1 \Rightarrow X_2) \to Y_1 \times (Y_1 \Rightarrow Y_2)$$

$$(x, g : Y_1 \to X_2) \mapsto (f_1(x), f_2 \circ g : Y_1 \to Y_2$$

$\searrow$ ~~$\cdot$~~ $GXf : X_2 \to Y_2$

is just f_2

$\nearrow$ $GfY : Y_2 \to Y_2$ is just identity