Finiteness Spaces and Monoidal Topology

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The Slogan

Working with finiteness spaces forces certain sums which would a priori be infinite to become finite, and hence well-defined, without resorting to limit structure. This will lead to a great many applications.

The first half of this talk contains results from papers with: R. Cockett, J. Beauvais-Feisthauer, I. Dewan, B. Drummond, P.-A. Jacqmin, P. Scott.

Ehrhard's finiteness spaces I

Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^\perp$ by:

$$\mathcal{U}^\perp = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^\perp \subseteq \mathcal{U}^\perp$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^\perp$

A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets.

Finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  (2) For all $v' \in \mathcal{F}(\mathbb{Y})^{\perp}$, we have $Rv' \in \mathcal{F}(\mathbb{X})^{\perp}$.

Composition is relational and it is straightforward to verify that this is a category. We denote it FinRel.

Lemma (Ehrhard)

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

$$(2')\ \textit{For all } b \in |\mathbb{Y}|, \textit{ we have } R\{b\} \in \mathcal{F}(\mathbb{X})^{\perp}.$$

Finiteness spaces III: It's a $*$-autonomous category

Theorem (Ehrhard)

FinRel *is a* $*$*-autonomous category. The tensor*

$$\mathbb{X} \otimes \mathbb{Y} = (|\mathbb{X} \otimes \mathbb{Y}|, \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}))$$

is given by setting $|\mathbb{X} \otimes \mathbb{Y}| = |\mathbb{X}| \times |\mathbb{Y}|$ *and*

$$\begin{aligned}\mathcal{F}(\mathbb{X} \otimes \mathbb{Y}) &= \{u \times v \mid u \in \mathcal{F}(\mathbb{X}), v \in \mathcal{F}(\mathbb{Y})\}^{\perp\perp} \\ &= \{w \mid \exists u \in \mathcal{F}(\mathbb{X}), \exists v \in \mathcal{F}(\mathbb{Y}), w \subseteq u \times v\}.\end{aligned}$$

In fact, FinRel is a model of all of linear logic (including exponentials).

Finiteness spaces IV: Other choices of morphism

Ehrhard was motivated by linear logic to construct a $*$-autonomous category and hence chose relations as morphisms. But the choice has issues. Much like the usual category of relations, FinRel is lacking many limits and colimits. Other choices are possible:

Definition

We define the category FinF*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a function satisfying the same conditions as above.*

Proposition

The category FinF *is a symmetric monoidal but not closed,. In particular, it doesn't have a terminal object.*

Finiteness spaces V: Other choices of morphism

Definition

We define the category FinPf*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

This fits perfectly with our intention of considering some topological groupoids as special finiteness spaces, since the multiplication is a partial map.

Ribenboim's generalized power series

- Ribenboim constructed rings of generalized power series for studies in number theory.
- While his construction gives a rich class of rings, it also seems ad hoc and non-functorial.
- We show that the conditions he imposes in fact can be used to construct internal monoids in a category of Ehrhard's *finiteness spaces* and the process is functorial.
- Furthermore any internal monoid of finiteness spaces induces a ring by Ehrhard's *linearization* process. So we get lots of new examples of generalized power series.

We'll need the following technical condition:
Let $(M, +, \leq)$ be a partially ordered (commutative) monoid. M is *strictly ordered* if

$$s < s' \Rightarrow s + t < s' + t \ \forall s, s', t \in M \ .$$

We will henceforth assume that all the monoids we work with are strictly ordered.

Definition

An ordered monoid is *artinian* if all strictly descending chains are finite; that is, if any list $(m_1 > m_2 > \cdots)$ must be finite. It is *narrow* if all discrete subsets are finite; that is, if all subsets of elements mutually unrelated by $\leq$ must be finite.

Ribenboim's generalized power series II

Definition

Let V be a vector space, and recall that the *support* of a function $f : M \longrightarrow V$ is defined by $supp(f) = \{m \in M | f(m) \neq 0\}$. Define the *space of Ribenboim power series from* M *with coefficients in* V, $G(M, V)$ to be the set of functions $f : M \longrightarrow V$ whose support is artinian and narrow.

If A is also a commutative $\mathbb{K}$-algebra, then $G(M, A)$ is a commutative $\mathbb{K}$-algebra with

$$(f \cdot g)(m) = \sum_{(u,v) \in X_m(f,g)} f(u) \cdot g(v)$$

where

$$X_m(f, g) := \{(u, v) \in M \times M | u + v = m \text{ and } f(u) \neq 0, g(v) \neq 0\}$$

Ribenboim's generalized power series III

This requires the following observation. It is where the strictness property gets used:

Proposition

The set $X_m(f,g)$ is finite for $f,g \in G(M,V)$.

This example is due to Ribenboim and was his motivation:

Let $M = \mathbb{N}\backslash\{0\}$ with the operation of multiplication, equipped with the usual ordering. Then $G(M,\mathbb{R})$ is the ring of arithmetic functions (i.e. functions from the positive integers to the complex numbers), and multiplication is Dirichlet's convolution:

$$(f \star g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$$

Posets as finiteness spaces I

Ribenboim's use of artinian and narrow subsets may seem unmotivated, but it in fact is precisely what we need to embed posets into finiteness spaces:

Theorem

Let $(P, \leq)$ be a poset. Let $\mathcal{U}$ be the set of artinian and narrow subsets. Then $(P, \mathcal{U})$ is a finiteness space.

Lemma

Under the above assumptions, $\mathcal{U}^{\perp}$ is the set of noetherian subsets of P.

Posets as finiteness spaces II: Functoriality

Unfortunately, if we consider the above construction from the usual category Pos of posets to any of the categories of finiteness spaces we have considered, it isn't functorial. Indeed, the inverse image under an order-preserving map of a noetherian subset may be not noetherian. However, the problem disappears if we consider *strict maps*.

Definition

If $(P, \leq)$ and $(Q, \leq)$ are two posets, a map $f\colon P \longrightarrow Q$ is said to be *strict* if $p < p'$ implies $f(p) < f(p')$. In particular, it is a morphism of posets. We denote the category of posets and strict maps by StrPos.

Proposition

The above construction is a strict symmetric monoidal functor $E\colon \mathrm{StrPos} \to \mathrm{FinPf}$.

Posets as finiteness spaces III: Internal monoids

As such, it takes monoids to monoids:

Theorem

The functor E induces a functor $Mon(E)\colon Mon(\mathrm{StrPos}) \to Mon(\mathrm{FinPf})$ from the category of strict pomonoids to the category of partial finiteness monoids.

Definition

A *partial finiteness monoid* is an internal monoid in FinPf.

Linearizing finiteness spaces and generalizing the Ribenboim construction

Let A be an abelian group and $\mathbb{X} = (X, \mathcal{U})$ a finiteness space. Ehrhard defined the abelian group $A\langle\mathbb{X}\rangle$ as the set

$$A\langle\mathbb{X}\rangle = \{f : X \to A \,|\, supp(f) \in \mathcal{U}\}$$

together with pointwise addition.

Lemma

In the case of a poset $(P, \leq)$ with its finiteness structure as determined as above, we recover $G(P, A)$.

Linearizing II

Theorem

If $(\mathbb{M}, \mu\colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \eta\colon I \to \mathbb{M})$ *is a partial finiteness monoid and* R *a ring (not necessarily commutative, but with unit), then* $R\langle\mathbb{M}\rangle$ *canonically has the structure of a ring.*
The multiplication in $R\langle\mathbb{M}\rangle$ *is given by*

$$(f \cdot g)(m) = \sum_{(m_1,m_2)\in X_m(f,g)} f(m_1) \cdot g(m_2).$$

Note the obvious similarity to Ribenboim's definition. But here it is the second condition in the definition of morphism of finiteness spaces that ensures the finiteness of the sum.

It's well-defined

Why is the set

$$X_m(f,g) = \{(m_1, m_2) \in M^2 \mid m_1 + m_2 = m,\ f(m_1) \neq 0,\ g(m_2) \neq 0\}$$

finite?

This set is exactly

$$\underbrace{(\mathrm{supp}(f) \times \mathrm{supp}(g))}_{\in \mathcal{W}} \cap \underbrace{\mu^{-1}(m)}_{\in \mathcal{W}^\perp}$$

Recall that μ is the multiplication. $\mathcal{W}$ is the finiteness space structure for $\mathbb{M} \otimes \mathbb{M}$.

The point

- This construction removes the ad hoc nature of the original construction.
- We obtain many new examples, including a construction by Newton, called *Puiseux series*.
- But the important thing here is it indicates that finiteness spaces can be used in situations where we need to force operations to be finite.

Brief digression: Higher-order category theory

All we are going to care about are a certain class of *bicategories* (Jean Benabou), i.e. we have 0-cells, 1-cells and 2-cells.

In fact, we're going to only consider a simple case of bicategories.

Definition

A *locally posetal bicategory* consists of an ordinary category such that each hom-set is a poset and the inequalities are preserved under composition.

Theorem

Rel, *the category of sets and relations, is a locally posetal bicategory. Each Hom-set is ordered under inclusion.*

The subject began with three classic papers.

- E. Manes. A triple theoretic construction of compact algebras, (1969).
- M. Barr. Relational algebras, (1970).
- B. Lawvere. Metric spaces, generalized logic, and closed categories, (1973).

There is a textbook on the subject now, which covers everything I'll say and lots more.

- D. Hofmann, G. Seal, W. Tholen (editors). **Monoidal Topology: A Categorical Approach to Order, Metric and Topology**, (2013).

Monoidal Topology I

- Ernie Manes demonstrated that the ultrafilter construction determines a monad on the category of sets and functions, and the category of Eilenberg-Moore algebras is equivalent to the category of compact hausdorff spaces.

- Barr showed that, roughly speaking, if we replace functions by relations in the above, and use the ordered structure of relations, we obtain arbitrary topological spaces.

Suppose we are in a locally posetal bicategory and I have a monad on that category. Then we can define *lax Eilenberg-Moore algebras* to be

$$
\begin{array}{ccc}
TTX & \xrightarrow{Ta} & TX \\
{\scriptstyle \mu}\downarrow & \geq & \downarrow{\scriptstyle a} \\
TX & \xrightarrow[a]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{e} & TX \\
 & \overset{\leq}{{\scriptstyle id}\searrow} & \downarrow{\scriptstyle a} \\
 & & X
\end{array}
$$

Theorem (Barr)

The category of lax Eilenberg-Moore algebras with respect to the ultrafilter monad is equivalent to the category of topological spaces.

Side Note

Note that we haven't defined morphisms between lax Eilenberg-Moore algebras. That will be easier to define when we talk about Q-categories.

Monoidal Topology III

Lawvere's observation was that, if we view a binary relation to be a function $X \times Y \to \text{Bool} = \{\top, \bot\}$, then we can write relational composition as:

$$R \colon X \to Y, S \colon Y \to Z \;\; \mapsto \quad R; S(x, z) = \bigvee_y R(x, y) \wedge S(y, z)$$

If we replace Bool with other structures, we can get generalized relations. As his main example, he chose $P^+ = ([0, \infty], \geq \text{(opposite of usual order)})$. He then used addition to model composition:

$$R \colon X \to Y, S \colon Y \to Z \;\; \mapsto \quad R; S(x, z) = \bigwedge_y R(x, y) + S(y, z)$$

Monoidal Topology IV

Then, if we consider the lax Eilenberg-Moore algebras for the identity monad, so if $a\colon X \to X$ is a lax algebra, we get the following inequalities:

$$a(x,y)+a(y,z) \geq a(x,z) \qquad 0 \geq a(x,x)(\text{which implies } 0 = a(x,x))$$

In other words, we get (pre)metric spaces. Both Lawvere and the authors of Monoidal Topology argue that these should be called metric spaces.

Monoidal Topology V

Similarly, if we take the identity monad and the usual Bool composition. We get the following:

$$a(x,y) \wedge a(y,z) \vdash a(x,z) \qquad \top \vdash a(x,x) (\text{which implies } \top = a(x,x))$$

So we get the usual axioms for a preorder. (Should we be calling these posets?)

We have the following "equations":

Identity monad + Bool truth values = posets.
Identity monad + $[0, \infty]$ truth values = metric spaces.
Ultrafilter monad + Bool truth values = topological spaces.
Ultrafilter monad + $[0, \infty]$ truth values = ???

Approach spaces (Robert Lowen)

Definition

An *approach space* is a set X equipped with a function $\delta \colon X \times P(X) \to [0, \infty]$ such that

- $\delta(x, \{x\}) = 0$
- $\delta(x, \emptyset) = \infty$
- $\delta(x, A \cup B) = min\{\delta(x, A), \delta(x, B)\}$
- For $u \in [0, \infty]$, $\delta(x, A) \leq \delta(x, A^{(u)}) + u$, where $A^{(u)} = \{x | \delta(x, A) \leq u\}$.

Definition

Let (X, δ) and (Y, δ') be approach spaces. A *morphism* of approach spaces is a function $f \colon X \to Y$ such that

$$\delta'(f(x), f(A)) \leq \delta(x, A)$$

Approach spaces II

This gives a category App. It is quite well-behaved.

- App has all products, coproducts, and arbitrary quotients.
- Every metric space is an approach space via $\delta(x, A) = min\{d(x, a) | a \in A\}$. The inclusion functor has a right adjoint.
- Every topological space is an approach space with

$$\delta(x, A) = \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{if not} \end{cases}$$

  The inclusion functor has a right adjoint.
- Non-metrizable spaces like the Stone–Cech compactification of the integers are approach spaces.

Approach spaces III

See *Approach spaces: the missing link in the topology-uniformity-metric triad* by Robert Lowen, (1997).

What's next?

We want to abstract these constructions. Abstractly, what can we take as our set of truth values? The best possible answer is *quantales*.

Some references:

- *Monoidal Topology*
- Joyal, Tierney. *An Extension of the Galois Theory of Grothendieck*, (1984).
- Rosenthal. *Quantales and Their Applications*, (1990).
- R. Blute, R. Kudzman-Blais, S. Niefield. *Constructing linear bicategories*, (2022).

Sup, the category of sup-lattices

Definition

A partially-ordered set which has arbitrary suprema is a *sup-lattice*.

Evidently, a poset with all sups also has all infs, but we call these sup-lattices because we want morphisms that preserve those.

Definition

The category Sup has sup-lattices as objects and sup-preserving maps as arrows.

Theorem

Sup *is a symmetric monoidal closed (in fact* $*$*-autonomous) category.*

The tensor product classifies maps that are sup-preserving in each variable precisely as the tensor product of vector spaces classifies maps that are linear in each variable.

Quantales

Definition (C. Mulvey)

A *quantale* is a monoid in Sup. Equivalently, it is a partially ordered set Q with all suprema and an associative multiplication $\otimes \colon Q \times Q \to Q$ with unit $\top$ such that for all subsets $P \subseteq Q$ and all elements $a \in Q$, we have

$$\left(\bigvee P\right) \otimes a = \bigvee_{p \in P} p \otimes a \quad \text{and} \quad a \otimes \left(\bigvee P\right) = \bigvee_{p \in P} a \otimes p$$

Note that Q necessarily satisfies $a \otimes \mathbf{0} = \mathbf{0} = \mathbf{0} \otimes a$, where $\mathbf{0}$ is the bottom element.

- We'll also assume an identity for the multiplication, which we'll denote by $\top$. Not all authors assume a unit.
- Bool and P^{+} are quantales.

Phase semantics

Girard's *phase semantics* of linear logic is an example of a quantale.

- If M is a monoid, $P(M)$ is a quantale, with multiplication defined for $A, B \in \mathcal{P}(M)$ by $A \otimes B = \{ab | a \in A, b \in B\}$.
- The $*$-autonomous version is a quantale as well. Let $\perp \subseteq M$ be arbitrary. Then if $A \subseteq M$, define

$$A^{\perp} = \{b \in M | \forall a \in A, ba \in M\}$$

.

Then a *fact* is a subset of M such that $A^{\perp\perp} = A$. Then the set of all facts forms a quantale with $A \hat{\otimes} B = (A \otimes B)^{\perp\perp}$

Examples From Rings

Let R be a ring. The following are examples of quantales. In each case, the multiplication is as follows. If $A, B \subseteq R$. Then

$$A \otimes B = \{a_0 b_0 + a_1 b_1 + \ldots + a_n b_n | a_i \in A, b_i \in B\}$$

- The set of additive subgroups of R
- The set of left ideals of R
- The set of right ideals of R
- The set of 2-sided ideals of R

Similarly, if one has a C^*-algebra, the sets of closed such subsets of the above form quantales.

Locales

See P. Johnstone, *Stone Spaces*.

Definition

A *locale, or frame,* is a quantale for which the associative operation is $\wedge$.

- Locales are the same thing as *complete Heyting algebras*.
- Let X be a topological space. Then $\mathcal{O}(X)$ is a locale. There is a construction in the other direction, assigning a space to each quantale, which is an adjunction. This is *pointless topology*.
- There is a notion of *measurable locale*, and one gets a category which is contravariantly equivalent to the category of commutative von Neumann algebras. This is *pointless measure theory*.

Q-Rel

If Q is a quantale, we can form the category Q-Rel whose objects are sets and arrows $f\colon X \to Y$ are functions $f\colon X \times Y \to Q$. Given $f\colon X \to Y$ and $g\colon Y \to Z$, composition is defined by

$$f \otimes g(x,z) = \bigvee_{y \in Y} f(x,y) \otimes g(y,z)$$

Note that the use of $\otimes$ on the left refers to composition and on the right refers to multiplication in Q.

Identities are given by

$$id_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases}$$

Lemma

If Q is a quantale, then Q-Rel is a locally posetal bicategory, under pointwise order.

Q-categories are categories enriched over Q, which is itself a monoidal category.

- Let Q be a quantale. An arrow $a\colon X \to X$ in Q-Rel is *transitive* if $a \otimes a \leq a$ and *reflexive* if $id_X \leq a$. A *Q-category* is a set equipped with a transitive, reflexive Q-relation. These conditions amount to:

$$a(x,y) \otimes a(y,z) \leq a(x,z) \qquad \top \leq a(x,x)$$

- If (X,a) and (Y,b) are Q-categories, then a Q-functor $f\colon (X,a) \to (Y,b)$ is a function $f\colon X \to Y$ such that

$$\forall x, x' \in X, a(x,x') \leq b(f(x), f(x'))$$

Examples of Q-categories

- If $Q = \text{Bool}$, then write $x \leq y$ for $a(x, y) = 1$, then a Bool-category is just an ordered set. Q-functors are order-preserving functions.

- If $Q = P^+$, then a Q-category is a metric space. Q-functors are non-expansive maps.

- If Q is any quantale, then Q is a Q-category with relation $\multimap$ defined by

$$m \multimap n = \bigvee \{q \in Q | m \otimes v \leq n\}$$

Then Q is a Q-category.

Morphisms

- Lax Eilenberg-Moore algebras with respect to the identity monad are the same thing as Q-categories.
- Then the notion of morphism corresponds to Q-functors. We pick up *order-preserving maps* and *non-expansive maps*.
- Similarly on can develop (Q, T)-categories where T is a monad, and this also gives the correct notion of morphism.

Linear bicategories [Cockett, Koslowski and Seely]

There is the following standard observation:

Monoidal categories are the same thing as 1-object bicategories.

We'd like to complete the following analogy:

Linearly distributive categories (models of $\otimes, \invamp$-fragment of linear logic) are the same thing as 1-object ??.

The answer will be *linear bicategories*.

Linear bicategories II

As usual with bicategories, one begins with a class of *0-cells* which we will denote $\mathcal{B}_0 = \{X, Y, Z, \ldots\}$. Then for every pair of 0-cells, one has a category $\mathcal{B}(X, Y)$. The objects of $\mathcal{B}(X, Y)$ are called *1-cells* and the arrows are called *2-cells*. But now we have two composition functors which we denote by $\otimes$ and $\oplus$:

$$\otimes, \oplus \colon \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \longrightarrow \mathcal{B}(X, Z)$$

such that each of these compositions gives a bicategory structure. Thus for each composition we have all of the morphisms and coherence that this entails.

These two bicategory structures are related by linear distributions as follows. Given:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow W$$

we have two functors:

$$-\otimes(-\parr-), (-\otimes-)\parr- : \mathcal{B}\,(X,Y)\times\mathcal{B}\,(Y,Z)\times\mathcal{B}\,(Z,W) \longrightarrow \mathcal{B}\,(X$$

and we require a natural transformation between them, which is not necessarily an isomorphism:

$$-\otimes(-\parr-) \Longrightarrow (-\otimes-)\parr-.$$

There are symmetric versions of this transformation as well as transformations involving the units.

Theorem (RB, Kudzman=Blais, Niefield)

*If Q is a Girard quantale, Q-*Rel *is a linear bicategory.*

Working with finiteness spaces eliminates the need to have the infinitary sups of quantales. What do we need instead?

Definition

A *partially ordered semiring* (or *po-semiring*) is a semiring $(R, +, \cdot, 0, 1)$ together with a partial order on the elements of R, such that

- $x \leq y$ implies $x + z \leq y + z$.
- $x \leq y, 0 \leq z$ imply $xz \leq yz$ and $zx \leq zy$.

A po-semiring is *positive* if every element is greater than or equal to 0. Positive po-semirings are extremely important in the theory of weighted automata among other places.

Adding order II

Positive po-semirings are extremely important in the theory of weighted automata among other places. See:

- M. Droste, W. Kuich. Semirings and Formal Power Series
- Droste, W. Kuich, H. Vogler. Handbook of Weighted Automata

Theorem

If R *is a positive po-semiring, then* Fin(R) *is a locally posetal bicategory, where hom-sets are ordered pointwise.*

We consider are two ordered semiring structures on the extended integers $\mathbb{Z}_{-} = \mathbb{Z} \cup \{-\infty\}$. (One could just as well consider the extended reals.)

- Multiplication is given by the usual addition of integers.
- Addition is given by the max-operation.
- This gives $\mathbb{Z}_{-}$ the structure of a positive partially ordered semiring with its usual order. This is the *arctic semiring*.
- Similarly one can consider the above, but with min as addition on $\mathbb{Z}_{\infty}$. This is a positive partially ordered semiring with the opposite of its usual order. This is the *tropical semiring*.

Idempotent mathematics

These semirings are hugely important in applied fields like formal language theory and synchronization for discrete event systems.

But there are lots of other positive partially ordered semirings about.

So the project:

Develop monoidal topology over finiteness spaces and positive partially ordered semirings. What kind of geometric structures do we get? What are the applications?