

# Topology for internal pre-orders: Focus on complete regularity

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(The talk is based in part on joint work with Amir Homayoun Nejah)

- The ultrafilter monad  $\mathcal{U}$
- The category  $\mathcal{U}\text{-Spa}$  and some of its subcategories
- Confessions
- Some other **Set**-monads
- The category  $T\text{-Spa}(\mathcal{C})$  and some of its subcategories
- The role of complete regularity in this context
- A glimpse at “elevated” ultrafilters

# The ultrafilter monad $\mathcal{U}$ on $\mathbf{Set}$ : what is it?

It is

- induced by adjunction

$$\mathbf{BoolAlg}^{\text{op}} \begin{array}{c} \xleftarrow{\text{hom}(-,2)} \\ \perp \\ \xrightarrow{\text{hom}(-,2)} \end{array} \mathbf{Set}$$

- induced by adjunction  $(\mathbf{Set}^{\text{End}(3)})^{\text{op}} \begin{array}{c} \xleftarrow{\text{hom}(-,3)} \\ \perp \\ \xrightarrow{\text{hom}(-,3)} \end{array} \mathbf{Set}$  (Lawvere 2000; Leinster 2013)

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- the codensity monad of  $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$  (Kennison-Gildenhuys 1971; Leinster 2013)

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# But what is it “really”?

$\mathfrak{x} \in \mathcal{U}X \iff \mathfrak{x}$  is a maximal down-directed and up-closed subset of  $\mathcal{P}X \setminus \{\emptyset\}$

Let  $f : X \rightarrow Y$ ,  $A \subseteq X$ ,  $B \subseteq Y$ ,  $x \in X$ ,  $\mathfrak{x} \in \mathcal{U}X$ ,  $\mathfrak{X} \in \mathcal{U}U\mathcal{X}$

$$uf : \mathcal{U}X \rightarrow \mathcal{U}Y \quad B \in \mathcal{U}f(\mathfrak{x}) \iff f^{-1}B \in \mathfrak{x}$$

$$(\dot{-}) : X \rightarrow \mathcal{U}X \quad A \in \dot{x} \iff x \in A$$

$$\Sigma : \mathcal{U}U\mathcal{X} \rightarrow \mathcal{U}\mathcal{X} \quad A \in \Sigma \mathfrak{X} \iff \{ \mathfrak{z} \in \mathcal{U}X \mid A \in \mathfrak{z} \} \in \mathfrak{X}$$

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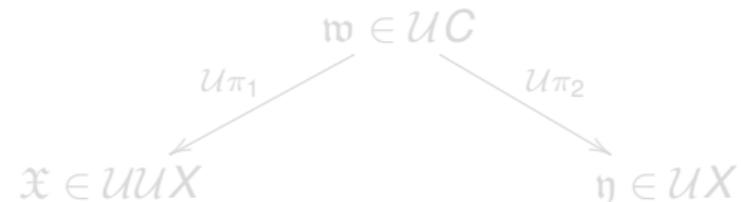
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# Ultrafilter convergence relation on $X$

$C \subseteq \mathcal{U}X \times X$ ; write  $\mathfrak{x} \rightsquigarrow y$  for  $(\mathfrak{x}, y) \in C$ ; axioms:

- (R)  $\dot{x} \rightsquigarrow x$
- (T)  $\underbrace{\mathfrak{x} \rightsquigarrow \mathfrak{y}}$  and  $\mathfrak{y} \rightsquigarrow z \implies \Sigma \mathfrak{x} \rightsquigarrow z$

$\exists \mathfrak{w} \in \mathcal{U}C (\mathcal{U}\pi_1(\mathfrak{w}) = \mathfrak{x}, \mathcal{U}\pi_2(\mathfrak{w}) = \mathfrak{y})$



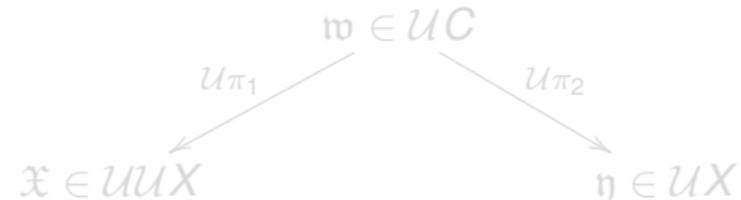
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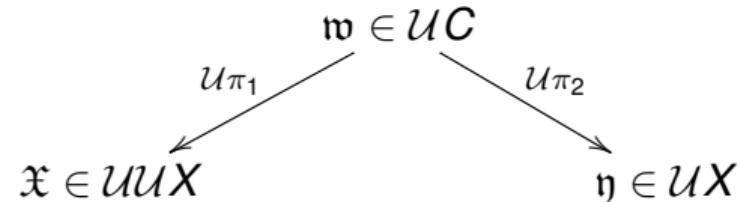
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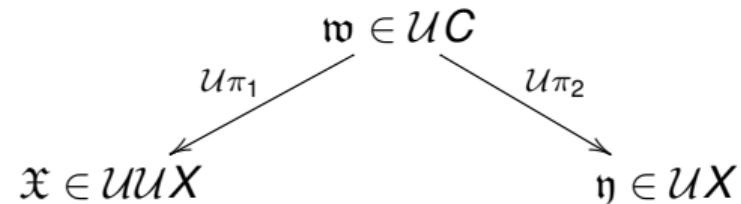
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# Just a picture: (R) and (T) for sequences

$$\dot{x} = (x, x, \dots) \quad \rightsquigarrow x$$

$$\mathfrak{x}_1 = (x_{1,1}, \dots) \quad \rightsquigarrow y_1$$

⋮

$$\mathfrak{x}_n = (x_{n,1}, \dots) \quad \rightsquigarrow y_n$$

⋮

$$\mathfrak{X} = (\mathfrak{x}_1, \dots) \quad \rightsquigarrow \mathfrak{y} \quad \rightsquigarrow Z \quad \Rightarrow \quad \Sigma \mathfrak{X} = (x_{1,1}, \dots, x_{n,n}, \dots) \quad \rightsquigarrow Z$$

# The category $\mathcal{U}\text{-}\mathbf{Spa}$

**$\mathcal{U}\text{-}\mathbf{Spa}$ :** objects are “ $\mathcal{U}$ -spaces”  $(X, C) = (X, \rightsquigarrow)$  satisfying (R) and (T)

= lax EM-algebras of ( $\hat{\mathcal{U}}$  = lax extension of  $\mathcal{U}$  from **Set** to **Rel**);

morphisms are  $\rightsquigarrow$ -preserving maps

$$\begin{array}{ccccc} X & \xrightarrow{(-)} & \mathcal{U}X & \xleftarrow{\hat{U}C} & \mathcal{U}\mathcal{U}X \\ & \searrow \subseteq & \downarrow C & \swarrow \subseteq & \downarrow \Sigma \\ & 1_X & & & \\ & & X & \xleftarrow{C} & \mathcal{U}X \end{array}$$

$$\begin{array}{ccc} \mathcal{U}X & \xrightarrow{\mathcal{U}f} & \mathcal{U}Y \\ \downarrow C_X & \subseteq & \downarrow C_Y \\ X & \xrightarrow{f} & Y \end{array}$$

# The forgetful functor $\mathcal{U}\text{-}\mathbf{Spa} \rightarrow \mathbf{Set}$

$\mathcal{U}\text{-}\mathbf{Spa} \rightarrow \mathbf{Set}$  is topological = all (discrete) “structured” cones admit cartesian liftings  
= fibration & cofibration & fibres are (large) complete

$$\begin{array}{ccc} (X, \rightsquigarrow) & \xrightarrow{f_i} & (Y_i, \rightsquigarrow_i) \\ \vdots & & \vdots \\ X & \xrightarrow{f_i} & Y_i \end{array} \qquad \qquad \mathfrak{x} \rightsquigarrow z \iff \forall i : \mathcal{U}f_i(\mathfrak{x}) \rightsquigarrow_i f_i(z)$$

Some consequences:

$\mathcal{U}\text{-}\mathbf{Spa}$  is (co)complete and (co)wellpowered, has (reg-epi, mono)-factorizations and (epi, reg-mono)-factorizations, etc., but fails to be cartesian closed.

Its underlying-set functor has both adjoints.

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# Some conditions defining full subcategories of $\mathcal{U}\text{-}\mathbf{Spa}$

Let  $X = (X, C) = (X, \rightsquigarrow)$  be a  $\mathcal{U}$ -space. Let's say that  $X$  is ...

- (A) if  $C$  is (the graph of) a map  $c: \forall x \in \mathcal{U}X \forall y \in X \quad (x \rightsquigarrow y \iff c(x) = y)$
- (K) if  $C$  is definable on all of  $\mathcal{U}X: \forall x \in \mathcal{U}X \exists y \in X \quad (x \rightsquigarrow y)$
- (H) if values of  $C$  are unique:  $\forall x \in \mathcal{U}X \forall y, z \in X \quad (x \rightsquigarrow y, x \rightsquigarrow z \implies y = z)$

Some easy, but important properties:

Consider objects  $X_i (i \in I)$  and morphisms  $f: X \rightarrow Y$  and  $f_i: X \rightarrow Y_i (i \in I)$  in  $\mathcal{U}\text{-}\mathbf{Spa}$ . Then:

- If  $f$  is epic and  $X$  is (K), so is  $Y$ .
- If every  $X_i$  is (K), so is  $\prod_{i \in I} X_i$ .
- If  $(f_i)_{i \in I}$  is jointly monic and every  $Y_i$  is (H), so is  $X$ .

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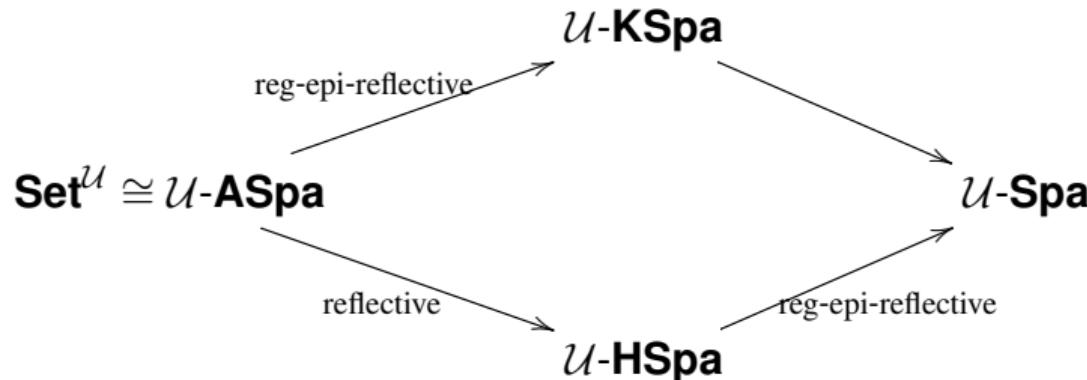
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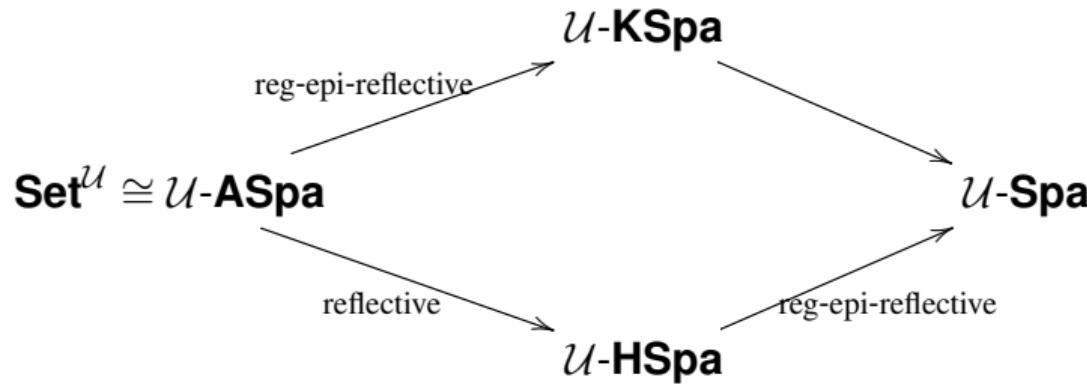
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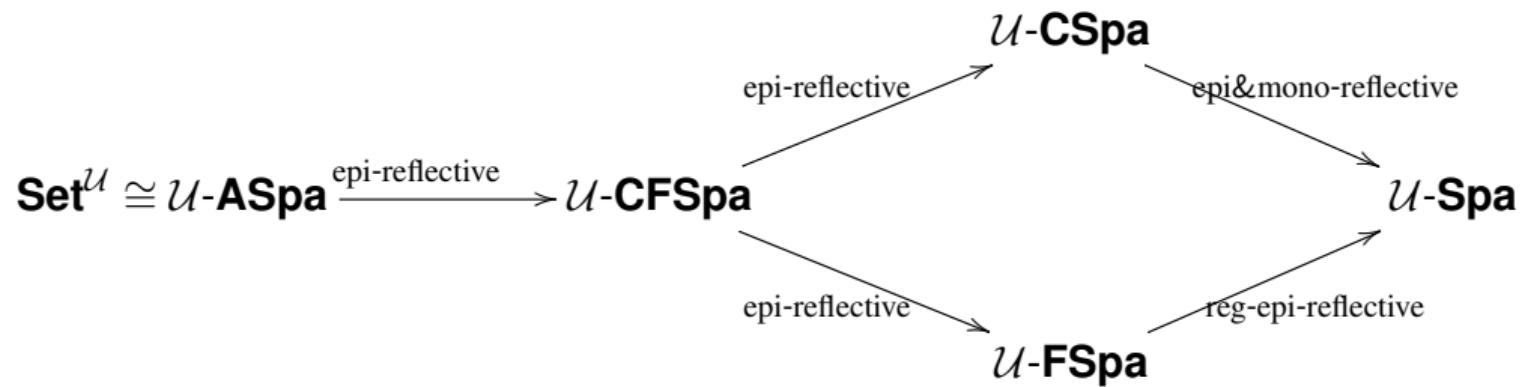
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(... see Möbus 1981 for further topological properties)

Moreover:

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- If “topology” refers exclusively to the geometry of neighbourhoods or open and closed sets, then we have not done any topology so far, just (quite elementary) category theory – even though we derived from it the Tychonoff Theorem, established the Stone-Čech-compactification, and several other reflections, etc.

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# Confessions

- $\mathbf{Set}^{\mathcal{U}} \cong \mathcal{U}\text{-}\mathbf{ASpa} \cong \mathbf{KHaus}$  (Manes 1967)
- $\mathcal{U}\text{-}\mathbf{Spa} \cong \mathbf{Top}$  (Barr 1970),  $\mathcal{U}\text{-}\mathbf{KSpa} \cong \mathbf{KTop}$ ,  $\mathcal{U}\text{-}\mathbf{HSpa} \cong \mathbf{Haus}$
- $\mathcal{U}\text{-}\mathbf{CSpa} \cong \mathbf{CReg}$  (Perry 1976),  $\mathcal{U}\text{-}\mathbf{FSpa} \cong \mathbf{FHaus}$ ,  $\mathcal{U}\text{-}\mathbf{CFSpa} \cong \mathbf{Tych}$ , ...  
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# Some examples

$T$	$T\text{-ASpa}$	$T\text{-Spa}$	$T\text{-CSpa}$
Id	<b>Set</b>	<b>Ord</b>	<b>Equ</b> (Burroni 1971)
$\mathcal{L}$	<b>Mon</b>	<b>MulOrd</b>	
$M \times (-)$	$\mathbf{Set}^M \cong \mathbf{Cat}_{\text{dcf}}/M$	$\mathbf{Lax}[M, \mathbf{Rel}] \cong \mathbf{Cat}_{\text{ff}}/M$	
$\mathcal{P}$	<b>Sup</b>	<b>PluOrd</b>	$\mathbf{Ord}_{\sup}$ ((Perry 1976))
$\mathcal{U}$	<b>KHaus</b>	<b>Top</b>	<b>CREg</b> (Perry 1976)

# A closer look at the power-set monad

The defining axioms (R) and (T) of a  $\mathcal{P}$ -space  $(X, \rightsquigarrow)$  may be stated as

$$(R) \{x\} \rightsquigarrow x \quad \text{and} \quad (T) (A_i \rightsquigarrow y_i \ (i \in I) \text{ and } \{y_i \mid i \in I\} \rightsquigarrow z) \implies \bigcup_{i \in I} A_i \rightsquigarrow z,$$

for all  $x, z, y_i \in X, A_i \subseteq X \ (i \in I)$ . The challenge now is to find a handy description of the universal quotient  $\mathcal{P}$ -homomorphism  $q$

$$X \xrightarrow{\{-\}} \mathcal{P}X \xrightarrow{q} Q$$

which makes the composite map  $\rightsquigarrow$ -preserving in a universal manner.

For  $(X, \rightsquigarrow)$  satisfying (C) one defines a preorder by  $(x \leq y \iff q(\{x\}) \leq q(\{y\}))$ , with respect to which one may describe the given “convergence relation”  $\rightsquigarrow$  by

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# A glimpse at $\mathcal{P}$ vs. $\mathcal{U}$

Consider the lax monad morphism  $\in: \hat{\mathcal{P}} \rightarrow \hat{\mathcal{U}}$  given by the membership relation  $A \in \mathfrak{x}$ :

$$\begin{array}{ccc}
 \mathcal{P}X & \xrightarrow{\hat{\mathcal{P}}R} & \mathcal{P}Y \\
 \epsilon_X \downarrow & \subseteq & \downarrow \epsilon_Y \\
 \mathcal{U}X & \xrightarrow{\hat{U}R} & \mathcal{U}Y
 \end{array}
 \quad
 \begin{array}{ccc}
 X & & \\
 \{\cdot\} \downarrow & = & \searrow \dot{(\cdot)} \\
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 \mathcal{P}\mathcal{P}X & \xrightarrow{\hat{\mathcal{P}}\epsilon_X} & \mathcal{P}\mathcal{U}X \xrightarrow{\epsilon_{\mathcal{U}X}} \mathcal{U}\mathcal{U}X \\
 U \downarrow & & \subseteq \qquad \qquad \downarrow \Sigma \\
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 \end{array}$$

It induces the “lax algebraic” functor

$$\mathbf{Top} = \mathcal{U}\text{-}\mathbf{Spa} \longrightarrow \mathcal{P}\text{-}\mathbf{Spa} = \mathbf{PluOrd}$$

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# Is **Set** relevant?

Hardly!

Consider any category  $\mathcal{C}$  equipped with a monad  $(T, \eta, \mu)$  such that (generously!)

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Modulo some adjustment of the vocabulary, we can now state:

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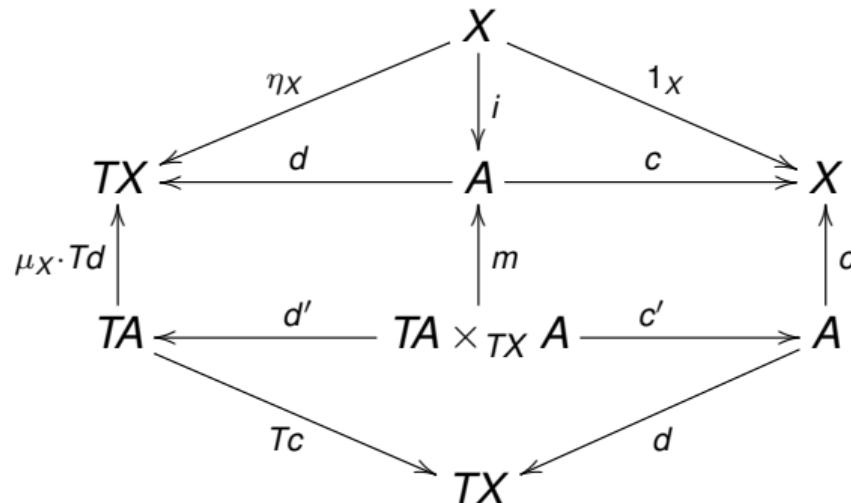
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# The category $T\text{-Spa}(\mathcal{C})$ : objects

Objects:  $(X, A, d, c)$  with  $(d, c) : A \rightarrow TX \times X$  monic and

(R)  $\exists i : X \rightarrow A (di = \eta_X, ci = 1_X)$  and (T)  $\exists m : A \rightarrow TA \times_{TX} A (dm = \mu_X(Td)d', cm = cc')$



Note: The morphisms  $i$  and  $m$  are uniquely determined.

# The category $T\text{-}\mathbf{Spa}(\mathcal{C})$ : morphisms

Morphisms:  $f : (X, A, d_X, c_X) \rightarrow (Y, B, d_Y, c_Y)$  are morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$  such that  $\exists \bar{f} : A \rightarrow B$  in  $\mathcal{C}$  ( $d_Y \cdot \bar{f} = (Tf)d_X$ ,  $c_Y \bar{f} = fc_X$ ) (which makes  $\bar{f}$  determined by  $f$ ).

$$\begin{array}{ccccc} TX & \xleftarrow{d_X} & A & \xrightarrow{c_X} & X \\ Tf \downarrow & & \downarrow \bar{f} & & \downarrow f \\ TY & \xleftarrow{d_Y} & B & \xrightarrow{c_Y} & Y \end{array}$$

## Theorem

The forgetful functor  $T\text{-}\mathbf{Spa}(\mathcal{C}) \rightarrow \mathcal{C}$ ,  $(X, A, d, c) \mapsto X$ , is topological. It therefore admits both, a left adjoint and a right adjoint. Like  $\mathcal{C}$ ,  $T\text{-}\mathbf{Spa}(\mathcal{C})$  is complete and well-powered and has a (regular epi, mono)-factorization system, which is preserved by the forgetful functor. Furthermore, if  $\mathcal{C}$  is cocomplete, so is  $T\text{-}\mathbf{Spa}(\mathcal{C})$ .

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# The full subcategories, Tychonoff and Stone-Čech

(A)	$d_X : A \rightarrow TX$ isomorphism	$T\text{-ASpa}(\mathcal{C}) \simeq \mathcal{C}^T$
(K)	$d_X : A \rightarrow TX$ split epimorphism*	$T\text{-KSpa}(\mathcal{C})$
(H)	$d_X : A \rightarrow TX$ monomorphism	$T\text{-HSpa}(\mathcal{C})$
(C)	$\beta_X : X \rightarrow BX$ cartesian (w.r.t. $T\text{-Spa}(\mathcal{C}) \rightarrow \mathcal{C}$ )	$T\text{-CSpa}(\mathcal{C})$
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\* Why not “regular epimorphism”, rather than “split epimorphism”?

Consider  $A$  of  $(X, A, d, c)$  as a morphism  $A : TX \rightarrow X$  in  $\text{Rel}(\mathcal{C})$ . Then:

$A$  is a *map* (à la Lawvere)  $\iff$  (K')  $\Delta_{TX} \leq A^\circ \circ A$  and (H')  $A \circ A^\circ \leq \Delta_X$

Now check: (K')  $\iff$  (K) and (H')  $\iff$  (H) !

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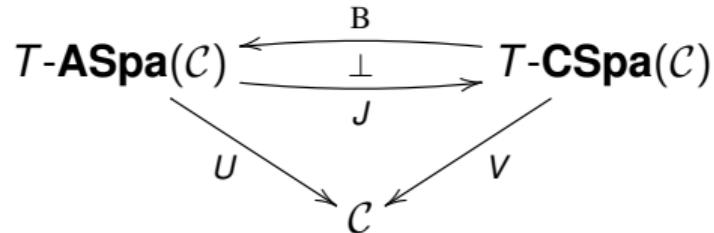
(1) Let  $X_i (i \in I)$ ,  $f : X \rightarrow Y$ , and  $f_i : X \rightarrow Y_i (i \in I)$  be in  $T\text{-}\mathbf{Spa}(\mathcal{C})$ . Then:

- If  $X$  is (K), so is  $Y$ , provided that (a)  $f$  is split epic OR (b)  $f$  regularly epic and  $Y$  is (H)
- (Tychonoff's Theorem) If every  $X_i$  is (K), so is  $\prod_{i \in I} X_i$ .
- If  $(f_i)_{i \in I}$  is jointly monic and every  $Y_i$  is (H), so is  $X$ .

(2) There is an adjunction  $T\text{-}\mathbf{ASpa}(\mathcal{C}) \rightleftarrows T\text{-}\mathbf{Spa}(\mathcal{C})$ .

$$\begin{array}{ccc} & \text{B} & \\ & \curvearrowleft & \\ T\text{-}\mathbf{ASpa}(\mathcal{C}) & \xrightleftharpoons[\text{J}]{\beta \perp 1} & T\text{-}\mathbf{Spa}(\mathcal{C}) \end{array}$$

# Considering $T\text{-ASpa}$ in $\text{CAT}/\mathcal{C}$

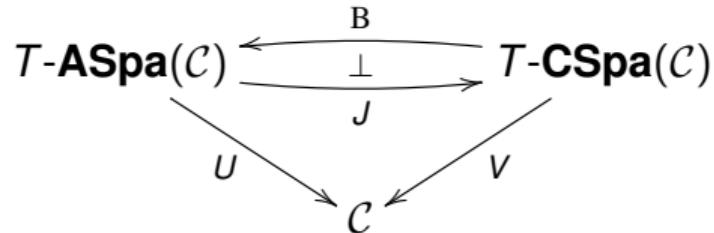


$J$  embeds the category  $T\text{-ASpa}$  reflectively into a fibred, even topological, category over  $\mathcal{C}$ .

How may this embedding be characterized?

A first answer was already given by Burroni in 1971, but some questions remained.

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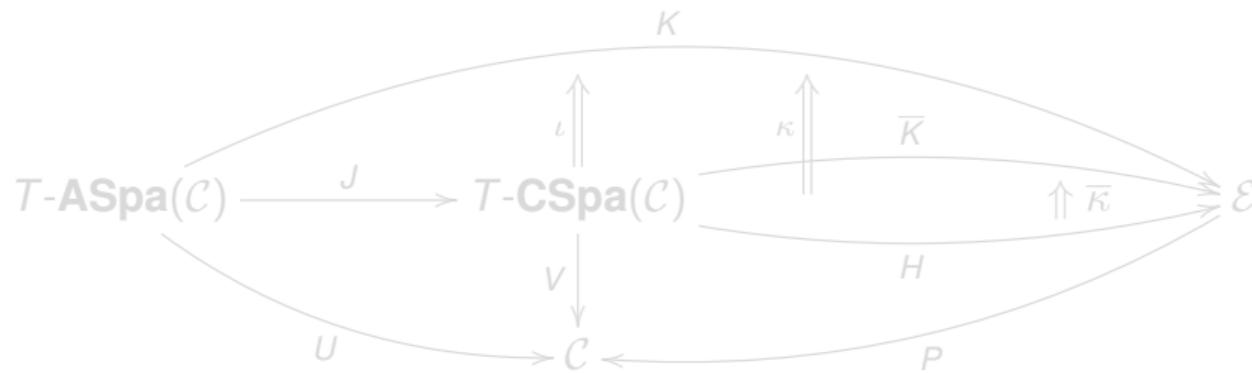
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# A universal property in $\mathbf{CAT}/\mathcal{C}$ -roughly

Key observation:  $\beta : \text{Id} \rightarrow JB$  is a pointwise  $V$ -cartesian lifting of  $V\beta : V \rightarrow VJB = UB$ .

Now we claim that the quadruple  $(J, V, B, \beta)$  is universal with this property:

Let  $K : T\text{-ASpa}(\mathcal{C}) \rightarrow \mathcal{E}$  and  $P : \mathcal{E} \rightarrow \mathcal{C}$  be functors, with  $PK = U = VJ$  being the forgetful functor, and assume that  $P$  is such that there is a pointwise  $P$ -cartesian lifting  $\vartheta : \overline{K} \rightarrow KB$  of  $V\beta : V \rightarrow UB = PKB$ .

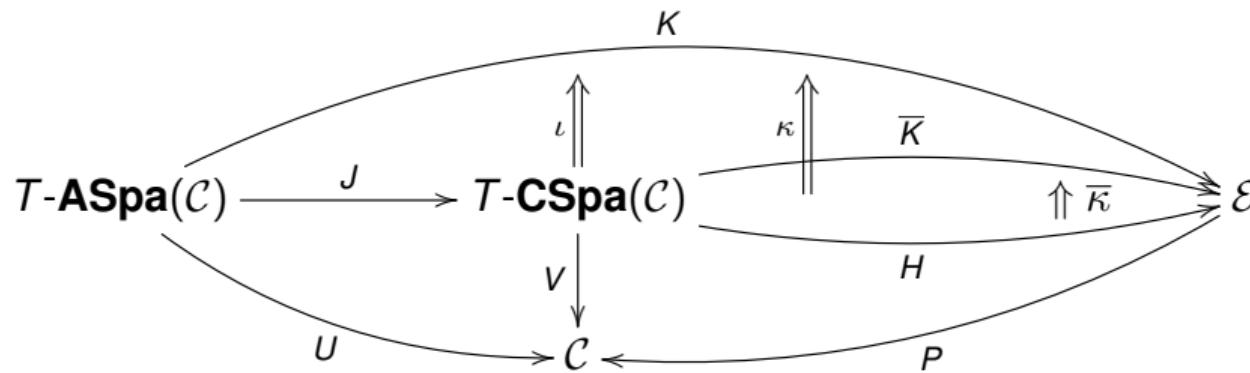


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# A “universal property” in $\mathbf{CAT}/\mathcal{C}$ –precisely

## Theorem

The functor  $\overline{K} : T\text{-}\mathbf{CSpa}(\mathcal{C}) \rightarrow \mathcal{E}$  satisfies the following properties (a),(b),(c); moreover, property (b) determines  $\overline{K}$  uniquely, while property (c) still determines  $\overline{K}$  up to a unique natural isomorphism whose  $P$ -image is an identity morphism:

- (a)  $P\overline{K} = V$ ;
- (b) there is a natural isomorphism  $\iota : \overline{K}J \rightarrow K$  with  $P\iota = 1_U$  and  $\iota B \cdot \overline{K}\beta = \vartheta$ ;
- (c)  $\overline{K}$  maps the pointwise  $V$ -cartesian transf.  $\beta$  to a pointwise  $P$ -cartesian lifting of  $V\beta$ .

Given any functor  $H : T\text{-}\mathbf{CSpa}(\mathcal{C}) \rightarrow \mathcal{E}$  with  $PH = V$  and a natural transformation  $\kappa : HJ \rightarrow K$  with  $P\kappa = 1_U$ , there is a unique natural transformation  $\overline{\kappa} : H \rightarrow \overline{K}$  with  $P\overline{\kappa} = 1_V$  and  $\iota \cdot \overline{\kappa}J = \kappa$ .

This means:  $\overline{K}$  and  $\iota$  form a *right Kan extension of  $K$  along  $J$* , not in  $\mathbf{CAT}$ , but in the 2-category  $\mathbf{CAT}/\mathcal{C}$  (where 2-cells, once mapped to the level of  $\mathcal{C}$ , are identities).



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This means:  $\overline{K}$  and  $\iota$  form a *right Kan extension of  $K$  along  $J$* , not in  $\mathbf{CAT}$ , but in the 2-category  $\mathbf{CAT}/\mathcal{C}$  (where 2-cells, once mapped to the level of  $\mathcal{C}$ , are identities).

# Short version of the Theorem

Corollary (Burroni 1971):

Let  $P : \mathcal{E} \rightarrow \mathcal{C}$  be a fibration. Then every functor  $K : T\text{-ASpa}(\mathcal{C}) \rightarrow \mathcal{E}$  with  $PK = U$  admits a pseudo-extension  $\bar{K} : T\text{-CSpa}(\mathcal{C}) \rightarrow \mathcal{E}$  over  $\mathcal{C}$  (so that  $\bar{K}J \cong K$  and  $P\bar{K} = V$ ) which, up to isomorphism, is uniquely determined by the property of mapping  $\beta$  to a pointwise  $P$ -cartesian natural transformation.

Note:

Burroni's paper actually claims that  $\bar{K}$  transforms *all*  $V$ -cartesian morphisms into  $P$ -cartesian morphisms—not just the  $V$ -cartesian morphisms  $\beta_X$  ( $X \in T\text{-CSpa}(\mathcal{C})$ ). We have not been able to confirm this claim and conjecture that it does not hold in general.

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The category  $\text{Gen}(\mathcal{C}^T)$  of  $T$  algebras with a system of generators:

$$\begin{array}{ccc}
 (TX, \mu_X) & \xrightarrow{f} & Y \\
 p^\sharp \downarrow & & \downarrow q \\
 (R, r) & \xrightarrow{f^*} & S
 \end{array}$$

The “codomain functor”  $C : \text{Gen}(\mathcal{C}^T) \rightarrow \mathcal{C}^T$  is left adjoint to the full embedding  $I : \mathcal{C}^T \rightarrow \text{Gen}(\mathcal{C}^T)$ ,  $(R, r) \mapsto (1_R, r)$ , with the adjunction units  $\gamma_{(p, r)} : (p, r) \rightarrow IC(p, r)$  :

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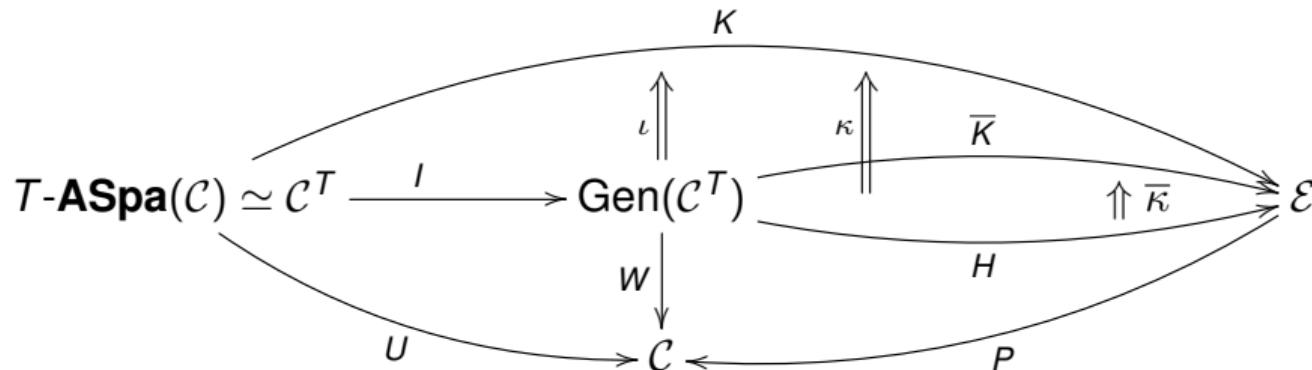
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... with the same (?) “universal property”

As a category over  $\mathcal{C}$ ,  $\text{Gen}(\mathcal{C}^T)$  behaves exactly like  $T\text{-CSpa}(\mathcal{C})$ ; that is:  
the previous theorem remains true *verbatim*—just trade  $(J, B, \beta, V)$  for  $(I, C, \gamma, W)$ :



# What's going on?

Let's look at our guiding example  $\mathcal{C} = \mathbf{Set}$ ,  $\mathcal{T} = \mathcal{U}$ :

The generalities established give us the diagram (commutative in both directions)

$$\begin{array}{ccccc} & & \mathcal{U}\text{-ASpa} \simeq \mathbf{Set}^{\mathcal{U}} \simeq \mathbf{K Haus} & & \\ & \swarrow J & & \searrow I & \\ \mathcal{U}\text{-CAlg} \simeq \mathbf{CReg} & & \bar{I} & & \text{Gen}(\mathbf{K Haus}) \\ & \searrow \perp & & \swarrow \bar{J} & \end{array}$$

- $\bar{I}$  maps the completely regular space  $X$  to the map  $\tilde{\beta}_X : X \rightarrow \tilde{B}X$ , where  $\tilde{B}X$  is the (closed) image of the map  $\beta_X^\sharp : \mathcal{U}X \rightarrow BX$ ,  $\mathfrak{x} \mapsto \lim \mathcal{U}\beta_X(\mathfrak{x})$ .
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# A conjecture

$$\begin{array}{ccc} T\text{-}\mathbf{ASpa}(\mathcal{C}) & \xrightarrow{K} & \mathcal{E} \\ J \downarrow & \nearrow \text{---} \bar{K} & \downarrow P \\ T\text{-}\mathbf{CSpa}(\mathcal{C}) & \xrightarrow{V} & \mathcal{C} \end{array}$$

You guess!

# “Elevated” ultrafilters

Which monads may lend themselves best for studying  $T\text{-}\mathbf{Spa}(\mathcal{C})$  and its subcategories?

- Already Manes (SLNM 80, 1969) studied “ $(T, \tilde{T})$ -bialgebras” over  $\mathbf{Set}$ , under some conditions on the interaction of the two participating monads, considering in particular the case  $T = M \times (-)$  for a monoid  $M$  and  $\tilde{T} = \mathcal{U}$ .
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Let us look at one of their examples, without alluding to its codensity status!

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# The simplified algebraic theory of $M$ -sets

Let  $M = (M, e, m)$  be a monoid.

$X$  is an  $M$ -set  $\iff X$  comes with a unital and associative action  $M \times X \rightarrow X$   
 $\iff X$  comes with a homomorphism  $M \rightarrow \mathbf{Set}(X, X)$  of monoids  
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Lawvere and Eilenberg-Moore in total harmony:  $\mathbf{Set}^M = [M, \mathbf{Set}]$

Actually,  $\mathbf{Set}$  may be replaced by *any* category  $\mathcal{C}$ ; define  $\mathcal{C}^M := [M, \mathcal{C}]$ .

For example:  $\mathbf{Top}^M$  is the category of  $M$ -sets that come equipped with a topology which makes all translation maps  $s : X \rightarrow X$  ( $s \in M$ ) continuous;  
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# The ultrafilter monad of $\mathbf{Set}^M$

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For an  $M$ -set  $X$ , make  $\mathbf{B}X$  an  $M$ -set, such that  $\beta_X : X \rightarrow \mathbf{B}X$  becomes equivariant, by:

$$s_{\mathfrak{z}} = (\mathbf{B}s)(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbf{B}X)$$

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Proof that this works and gives the desired adjunction: **CAT** is a 2-category!

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Now it is easy to see that

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# More examples?

Wait and see!

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