Topology for internal pre-orders: Focus on complete regularity

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(The talk is based in part on joint work with Amir Homayoun Nejah)

Overview

- The ultrafilter monad $\mathcal{U}$
- The category $\mathcal{U}$-Spa and some of its subcategories
- Confessions
- Some other **Set**-monads
- The category T-Spa$(\mathcal{C})$ and some of its subcategories
- The role of complete regularity in this context
- A glimpse at “elevated” ultrafilters

The ultrafilter monad $\mathcal{U}$ on **Set**: what is it?

It is

- induced by adjunction $\begin{tikzcd} \mathbf{BooAlg}^{\mathrm{op}} \arrow[r, bend right, "\hom(-{,}2)"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "\hom(-{,}2)"'] \end{tikzcd}$
- induced by adjunction $\begin{tikzcd} (\mathbf{Set}^{\mathrm{End}(3)})^{\mathrm{op}} \arrow[r, bend right, "\hom(-{,}3)"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "\hom(-{,}3)"'] \end{tikzcd}$ (Lawvere 2000; Leinster 2013)
- induced by the adjunction $\begin{tikzcd} \mathbf{KHaus} \arrow[r, bend right, "\mathrm{forget}"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "\beta"'] \end{tikzcd}$
- the codensity monad of **FinSet** $\hookrightarrow$ **Set** (Kennison-Gildenhuys 1971; Leinster 2013)
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But what is it “really”?

$\mathfrak{x} \in \mathcal{U}X \iff \mathfrak{x}$ is a maximal down-directed and up-closed subset of $\mathcal{P}X \setminus \{\emptyset\}$

Let $f : X \longrightarrow Y, \quad A \subseteq X, \quad B \subseteq Y, \quad x \in X, \quad \mathfrak{x} \in \mathcal{U}X, \quad \mathfrak{X} \in \mathcal{U}\mathcal{U}X$

$$\mathcal{U}f : \mathcal{U}X \longrightarrow \mathcal{U}Y \qquad B \in \mathcal{U}f(\mathfrak{x}) \iff f^{-1}B \in \mathfrak{x}$$

$$(\dot{-}) : X \longrightarrow \mathcal{U}X \qquad A \in \dot{x} \iff x \in A$$

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Ultrafilter convergence relation on X

$C \subseteq \mathcal{U}X \times X$; write $\mathfrak{x} \rightsquigarrow y$ for $(\mathfrak{x}, y) \in C$; axioms:

- (R) $\dot{x} \rightsquigarrow x$
- (T) $\underbrace{\mathfrak{X} \mathrel{\hat{\rightsquigarrow}} \mathfrak{y}}$ and $\mathfrak{y} \rightsquigarrow z \implies \Sigma\mathfrak{X} \rightsquigarrow z$

$\exists\ \mathfrak{w} \in \mathcal{U}C\ (\mathcal{U}\pi_1(\mathfrak{w}) = \mathfrak{X},\ \mathcal{U}\pi_2(\mathfrak{w}) = \mathfrak{y})$

$$\begin{tikzcd} & \mathfrak{w} \in \mathcal{U}C \arrow[dl, "\mathcal{U}\pi_1"'] \arrow[dr, "\mathcal{U}\pi_2"] & \\ \mathfrak{X} \in \mathcal{U}\mathcal{U}X & & \mathfrak{y} \in \mathcal{U}X \end{tikzcd}$$

We extended C to $\hat{\mathcal{U}}C = \hat{C} \subseteq \mathcal{U}\mathcal{U}X \times \mathcal{U}X$, writing $\mathfrak{X} \mathrel{\hat{\rightsquigarrow}} \mathfrak{y}$ for $(\mathfrak{X}, \mathfrak{y}) \in \hat{C}$.

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Just a picture: (R) and (T) for sequences

$$\dot{x} = (x, x, ...) \quad \rightsquigarrow \quad x$$

$$\mathfrak{x}_1 = (x_{1,1}, ...) \quad \rightsquigarrow \quad y_1$$

$$\vdots \qquad \vdots$$

$$\mathfrak{x}_n = (x_{n,1}, ...) \quad \rightsquigarrow \quad y_n$$

$$\vdots \qquad \vdots$$

$$\mathfrak{X} = (\mathfrak{x}_1, ...) \quad \rightsquigarrow \quad \mathfrak{y} \quad \rightsquigarrow z \quad \Rightarrow \quad \Sigma\mathfrak{X} = (x_{1,1}, .., x_{n,n}, ...) \quad \rightsquigarrow \quad z$$

The category $\mathcal{U}$-**Spa**

$\mathcal{U}$-**Spa**: objects are "$\mathcal{U}$-spaces" $(X, C) = (X, \rightsquigarrow)$ satisfying (R) and (T)

= lax EM-algebras of ($\hat{\mathcal{U}}$ = lax extension of $\mathcal{U}$ from **Set** to **Rel**);

morphisms are $\rightsquigarrow$-preserving maps

$$\begin{tikzcd}
X \arrow[r, "(\dot{-})"] \arrow[dr, "1_X"'] & \mathcal{U}X \arrow[d, "c"] \arrow[dl, phantom, "\subseteq"] & & \mathcal{U}\mathcal{U}X \arrow[ll, "\hat{\mathcal{U}}c"'] \arrow[d, "\Sigma"] \arrow[dll, phantom, "\subseteq"] \\
& X & & \mathcal{U}X \arrow[ll, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U}X \arrow[r, "\mathcal{U}f"] \arrow[d, "c_X"'] \arrow[dr, phantom, "\subseteq"] & \mathcal{U}Y \arrow[d, "c_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The forgetful functor $\mathcal{U}$-**Spa** $\longrightarrow$ **Set**

$\mathcal{U}$-**Spa** $\longrightarrow$ **Set** is topological = all (discrete) "structured" cones admit cartesian liftings
= fibration & cofibration & fibres are (large) complete

$$\begin{tikzcd}
(X, \rightsquigarrow) \arrow[r, "f_i"] \arrow[d, dotted, no head] & (Y_i, \rightsquigarrow_i) \arrow[d, dotted, no head] \\
X \arrow[r, "f_i"] & Y_i
\end{tikzcd}$$

$$\mathfrak{x} \rightsquigarrow z \iff \forall i : \mathcal{U} f_i(\mathfrak{x}) \rightsquigarrow_i f_i(z)$$

Some consequences:

$\mathcal{U}$-**Spa** is (co)complete and (co)wellpowered, has (reg-epi, mono)-factorizations and (epi, reg-mono)-factorizations, *etc.*, but fails to be cartesian closed.

Its underlying-set functor has both adjoints.

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Some conditions defining full subcategories of $\mathcal{U}$-**Spa**

Let $X = (X, C) = (X, \rightsquigarrow)$ be a $\mathcal{U}$-space. Let's say that X is ...

(A)	if C is (the graph of) a map c:	$\forall\, \mathfrak{x} \in \mathcal{U}X\ \forall\, y \in X$	$(\, \mathfrak{x} \rightsquigarrow y \iff c(\mathfrak{x}) = y\,)$
(K)	if C is definable on all of $\mathcal{U}X$:	$\forall\, \mathfrak{x} \in \mathcal{U}X\ \exists\, y \in X$	$(\, \mathfrak{x} \rightsquigarrow y\,)$
(H)	if values of C are unique:	$\forall\, \mathfrak{x} \in \mathcal{U}X\ \forall\, y, z \in X$	$(\mathfrak{x} \rightsquigarrow y,\ \mathfrak{x} \rightsquigarrow z \Longrightarrow y = z)$

Some easy, but important properties:

Consider objects $X_i\ (i \in I)$ and morphisms $f : X \to Y$ and $f_i : X \to Y_i\ (i \in I)$ in $\mathcal{U}$-**Spa**. Then:

- If f is epic and X is (K), so is Y.
- If every X_i is (K), so is $\prod_{i \in I} X_i$.
- If $(f_i)_{i \in I}$ is jointly monic and every Y_i is (H), so is X.

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Easy properties + GAFT-methods give reflections

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\begin{tikzcd}
 & \mathcal{U}\text{-}\mathbf{KSpa} \arrow[dr] & \\
\mathbf{Set}^{\mathcal{U}} \cong \mathcal{U}\text{-}\mathbf{ASpa} \arrow[ur, "\text{reg-epi-reflective}"] \arrow[dr, "\text{reflective}"'] & & \mathcal{U}\text{-}\mathbf{Spa} \\
 & \mathcal{U}\text{-}\mathbf{HSpa} \arrow[ur, "\text{reg-epi-reflective}"'] &
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In particular:

$\mathcal{U}$-**ASpa** $\hookrightarrow$ $\mathcal{U}$-**Spa** is reflective, with adjunction units

$$\beta_X = (X \xrightarrow{(\dot{-})} \mathcal{U}X \xrightarrow{q} \mathrm{B}X)$$

and q the quotient map given by the least congruence relation making β_X a morphism.

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(C) if $\beta_X : X \to \mathrm{B}X$ is cartesian w.r.t. $\mathcal{U}$-**Spa**$\longrightarrow$ **Set**;

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Easy, but important properties: Consider $f_i : X \to Y_i$ $(i \in I)$ in $\mathcal{U}$-**Spa**. Then:

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 & & \mathcal{U}\text{-}\mathbf{CSpa} \arrow[dr, "\text{epi\&mono-reflective}"] & \\
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Confessions

- $\mathbf{Set}^{\mathcal{U}} \cong \mathcal{U}$-**ASpa** $\cong$ **KHaus** (Manes 1967)
- $\mathcal{U}$-**Spa** $\cong$ **Top** (Barr 1970), $\mathcal{U}$-**KSpa** $\cong$ **KTop**, $\mathcal{U}$-**HSpa** $\cong$ **Haus**
- $\mathcal{U}$-**CSpa** $\cong$ **CReg** (Perry 1976), $\mathcal{U}$-**FSpa** $\cong$ **FHaus**, $\mathcal{U}$-**CFSpa** $\cong$ **Tych**, ...
  (... see Möbus 1981 for further topological properties)

Moreover:

- In all of the above, we never used any particular feature of $\mathcal{U}$, other than the fact that $\mathcal{U}$ is a monad on **Set** . In other words:
- EVERYTHING said on the previous slides remains valid for ANY monad T on **Set**! (Just replace $\mathcal{U}$ by T and avoid the word "ultrafilter".) Or:
- If "topology" refers exclusively to the geometry of neighbourhoods or open and closed sets, then we have not done any topology so far, just (quite elementary) category theory – even though we derived from it the Tychonoff Theorem, established the Stone-Čech-compactification, and several other reflections, *etc.*

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- $\mathbf{Set}^{\mathcal{U}} \cong \mathcal{U}$-**ASpa** $\cong$ **KHaus** (Manes 1967)
- $\mathcal{U}$-**Spa** $\cong$ **Top** (Barr 1970), $\mathcal{U}$-**KSpa** $\cong$ **KTop**, $\mathcal{U}$-**HSpa** $\cong$ **Haus**
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Some examples

T	T-**ASpa**	T-**Spa**	T-**CSpa**	
Id	**Set**	**Ord**	**Equ**	(Burroni 1971)
$\mathcal{L}$	**Mon**	**MulOrd**		
$M\times(\text{-})$	$\mathbf{Set}^M \cong \mathbf{Cat}_{\mathrm{dcf}}/M$	$\mathrm{Lax}[M, \mathbf{Rel}] \cong \mathbf{Cat}_{\mathrm{ff}}/M$		
$\mathcal{P}$	**Sup**	**PluOrd**	$\mathbf{Ord}_{\mathrm{sup}}$	((Perry 1976))
$\mathcal{U}$	**KHaus**	**Top**	**CReg**	(Perry 1976)

A closer look at the power-set monad

The defining axioms (R) and (T) of a $\mathcal{P}$-space $(X, \rightsquigarrow)$ may be stated as

$$\text{(R) } \{x\} \rightsquigarrow x \quad \text{and} \quad \text{(T) } (A_i \rightsquigarrow y_i \ (i \in I) \text{ and } \{y_i \mid i \in I\} \rightsquigarrow z) \Longrightarrow \bigcup_{i \in I} A_i \rightsquigarrow z,$$

for all $x, z, y_i \in X, A_i \subseteq X \ (i \in I)$. The challenge now is to find a handy description of the universal quotient $\mathcal{P}$-homomorphism q

$$X \xrightarrow{\{-\}} \mathcal{P}X \xrightarrow{q} \mathrm{Q}$$

which makes the composite map $\rightsquigarrow$-preserving in a universal manner.
For $(X, \rightsquigarrow)$ satisfying (C) one defines a preorder by $(x \leq y \iff q(\{x\}) \leq q(\{y\}))$, with respect to which one may describe the given "convergence relation" $\rightsquigarrow$ by

$$A \rightsquigarrow y \iff \sup A \text{ exists in } X \text{ and } \sup A \simeq y$$

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A glimpse at $\mathcal{P}$ vs. $\mathcal{U}$

Consider the lax monad morphism $\in : \hat{\mathcal{P}} \to \hat{\mathcal{U}}$ given by the membership relation $A \in \mathfrak{x}$:

$$
\begin{tikzcd}
\mathcal{P}X \arrow[r, "\hat{\mathcal{P}}R"] \arrow[d, "\in_X"'] \arrow[dr, phantom, "\subseteq"] & \mathcal{P}Y \arrow[d, "\in_Y"] \\
\mathcal{U}X \arrow[r, "\hat{\mathcal{U}}R"'] & \mathcal{U}Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\{-\}"'] \arrow[dr, "(\dot{-})"] \arrow[dr, phantom, "=", pos=0.3] & \\
\mathcal{P}X \arrow[r, "\in_X"'] & \mathcal{U}X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}\mathcal{P}X \arrow[r, "\hat{\mathcal{P}}\in_X"] \arrow[d, "\bigcup"'] & \mathcal{P}\mathcal{U}X \arrow[r, "\in_{\mathcal{U}X}"] \arrow[d, phantom, "\subseteq"] & \mathcal{U}\mathcal{U}X \arrow[d, "\Sigma"] \\
\mathcal{P}X \arrow[rr, "\in_X"'] & & \mathcal{U}X
\end{tikzcd}
$$

It induces the "lax algebraic" functor

$$\mathbf{Top} = \mathcal{U}\text{-}\mathbf{Spa} \longrightarrow \mathcal{P}\text{-}\mathbf{Spa} = \mathbf{PluOrd}$$

sending $(X, \to)$ to $(X, \rightsquigarrow)$ with $\begin{tikzcd} \mathcal{P}X \arrow[r, "\in"'] \arrow[rr, bend left, "\rightsquigarrow"] & \mathcal{U}X \arrow[r, "\to"'] & X \end{tikzcd}$; that is:

$$A \rightsquigarrow y \iff \exists \mathfrak{x} \in \mathcal{U}X\,(A \in \mathfrak{x} \text{ and } \mathfrak{x} \to y) \iff y \in \overline{A}\,.$$

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Is **Set** relevant?

Hardly!

Consider any category $\mathcal{C}$ equipped with a monad (T, η, μ) such that (generously!)

- $\mathcal{C}$ is complete and well-powered;
- all morphisms in $\mathcal{C}$ factor (regular epi, mono);
- $T : \mathcal{C} \to \mathcal{C}$ preserves regular epimorphisms.

Modulo some adjustment of the vocabulary, we can now state:

Theorem

When we trade $(\mathbf{Set}, \mathcal{U})$ for $(\mathcal{C}, T)$, everything said before the Confession survives.

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Objects: (X, A, d, c) with $(d, c) : A \to TX \times X$ monic and

(R) $\exists i : X \to A\,(di = \eta_X, ci = 1_X)$ and (T) $\exists m : A \to TA \times_{TX} A\,(dm = \mu_X(Td)d', cm = cc')$

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[d, "i"] \arrow[dr, "1_X"] & \\
TX & A \arrow[l, "d"'] \arrow[r, "c"] & X \\
TA \arrow[u, "\mu_X \cdot Td"] \arrow[dr, "Tc"'] & TA \times_{TX} A \arrow[l, "d'"'] \arrow[r, "c'"] \arrow[u, "m"'] & A \arrow[u, "c"'] \arrow[dl, "d"] \\
 & TX &
\end{tikzcd}$$

Note: The morphisms i and m are uniquely determined.

The category T-Spa($\mathcal{C}$): morphisms

Morphisms: $f : (X, A, d_X, c_X) \longrightarrow (Y, B, d_Y, c_Y)$ are morphisms $f : X \to Y$ in $\mathcal{C}$ such that $\exists\, \bar{f} : A \to B$ in $\mathcal{C}$ $\;(d_Y \cdot \bar{f} = (Tf)d_X,\;\; c_Y\bar{f} = fc_X)$ (which makes $\bar{f}$ determined by f).

$$\begin{tikzcd} TX \arrow[d, "Tf"'] & A \arrow[l, "d_X"'] \arrow[r, "c_X"] \arrow[d, "\bar{f}"] & X \arrow[d, "f"] \\ TY & B \arrow[l, "d_Y"] \arrow[r, "c_Y"'] & Y \end{tikzcd}$$

Theorem

*The forgetful functor T-**Spa**$(\mathcal{C}) \longrightarrow \mathcal{C}$, $(X, A, d, c) \mapsto X$, is topological. It therefore admits both, a left adjoint and a right adjoint. Like $\mathcal{C}$, T-**Spa**$(\mathcal{C})$ is complete and well-powered and has a (regular epi, mono)-factorization system, which is preserved by the forgetful functor. Furthermore, if $\mathcal{C}$ is cocomplete, so is T-**Spa**$(\mathcal{C})$.*

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The full subcategories, Tychonoff and Stone-Čech

(A)	$d_X : A \to TX$ isomorphism	T-**ASpa**$(\mathcal{C}) \simeq \mathcal{C}^T$
(K)	$d_X : A \to TX$ split epimorphism*	T-**KSpa**$(\mathcal{C})$
(H)	$d_X : A \to TX$ monomorphism	T-**HSpa**$(\mathcal{C})$
(C)	$\beta_X : X \to \mathrm{B}X$ cartesian (w.r.t. T-**Spa**$(\mathcal{C}) \longrightarrow \mathcal{C}$)	T-**CSpa**$(\mathcal{C})$
(F)	$\beta_X : X \to \mathrm{B}X$ monomorphism	T-**FSpa**$(\mathcal{C})$

* Why not "regular epimorphism", rather than "split epimorphism"?

Consider A of (X, A, d, c) as a morphism $A : TX \to X$ in **Rel**$(\mathcal{C})$. Then:

A is a *map* (à la Lawvere) $\iff$ (K') $\Delta_{TX} \le A^\circ \circ A$ and (H') $A \circ A^\circ \le \Delta_X$

Now check: (K') $\iff$ (K) and (H') $\iff$ (H) !

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Tychonoff and Stone-Čech

Theorem

(1) *Let* $X_i\,(i \in I)$, $f : X \to Y$, *and* $f_i : X \to Y_i\,(i \in I)$ *be in* T-**Spa**$(\mathcal{C})$. *Then:*

- *If* X *is* (K), *so is* Y, *provided that* (a) f *is split epic OR* (b) f *regularly epic and* Y *is* (H)
- (Tychonoff's Theorem) *If every* X_i *is* (K), *so is* $\prod_{i \in I} X_i$.
- *If* $(f_i)_{i \in I}$ *is jointly monic and every* Y_i *is* (H), *so is* X.

(2) *There is an adjunction*

$$
\begin{tikzcd}
T\text{-}\mathbf{ASpa}(\mathcal{C}) \arrow[r, hook, "J"'] \arrow[r, phantom, shift left=2, "\beta \perp 1"] & T\text{-}\mathbf{Spa}(\mathcal{C}) \arrow[l, bend right, "\mathrm{B}"']
\end{tikzcd} .
$$

Considering T-**CSpa** in **CAT**/$\mathcal{C}$

$$\begin{tikzcd}
T\text{-}\mathbf{ASpa}(\mathcal{C}) \arrow[rr, bend right, "J"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "U"'] & & T\text{-}\mathbf{CSpa}(\mathcal{C}) \arrow[ll, bend right, "\mathrm{B}"'] \arrow[dl, "V"] \\
& \mathcal{C} &
\end{tikzcd}$$

J embeds the category T-**ASpa** reflectively into a fibred, even topological, cateory over $\mathcal{C}$.

How may this embedding be characterized?

A first answer was already given by Burroni in 1971, but some questions remained.

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A universal property in **CAT**/$\mathcal{C}$–roughly

Key observation: $\beta : \mathrm{Id} \to J\mathrm{B}$ is a pointwise V-cartesian lifting of $V\beta : V \to VJ\mathrm{B} = U\mathrm{B}$.

Now we claim that the quadruple $(J, V, \mathrm{B}, \beta)$ is universal with this property:

Let $K : T\text{-}\mathbf{ASpa}(\mathcal{C}) \to \mathcal{E}$ and $P : \mathcal{E} \to \mathcal{C}$ be functors, with $PK = U = VJ$ being the forgetful functor, and assume that P is such that there is a pointwise P-cartesian lifting $\vartheta : \overline{K} \to K\mathrm{B}$ of $V\beta : V \to U\mathrm{B} = PK\mathrm{B}$.

$$\begin{tikzcd}
T\text{-}\mathbf{ASpa}(\mathcal{C}) \arrow[r, "J"] \arrow[rr, bend left=50, "K"] \arrow[r, phantom, bend left=30, "\Uparrow\iota"] \arrow[dr, bend right, "U"'] & T\text{-}\mathbf{CSpa}(\mathcal{C}) \arrow[r, bend left, "\overline{K}"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow\overline{\kappa}"] \arrow[r, phantom, bend left=40, "\Uparrow\kappa"] \arrow[d, "V"] & \mathcal{E} \arrow[dl, bend left, "P"] \\
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T\text{-}\mathbf{ASpa}(\mathcal{C}) \arrow[r, "J"] \arrow[rr, bend left=40, "K"] \arrow[rr, phantom, bend left=20, "\Uparrow \iota"] \arrow[dr, bend right, "U"'] & T\text{-}\mathbf{CSpa}(\mathcal{C}) \arrow[r, bend left, "\overline{K}"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow \overline{\kappa}"] \arrow[r, phantom, bend left=40, "\Uparrow \kappa"] \arrow[d, "V"'] & \mathcal{E} \arrow[dl, bend left, "P"] \\
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A "universal property" in **CAT**/$\mathcal{C}$–precisely

Theorem

The functor $\overline{K} : T$-**CSpa**$(\mathcal{C}) \to \mathcal{E}$ *satisfies the following properties* (a),(b),(c)*; moreover, property* (b) *determines* $\overline{K}$ *uniquely, while property* (c) *still determines* $\overline{K}$ *up to a unique natural isomorphism whose* P*-image is an identity morphism:*

(a) $P\overline{K} = V$;

(b) *there is a natural isomorphism* $\iota : \overline{K}J \to K$ *with* $P\iota = 1_U$ *and* $\iota\mathrm{B} \cdot \overline{K}\beta = \vartheta$;

(c) $\overline{K}$ *maps the pointwise* V*-cartesian transf.* β *to a pointwise* P*-cartesian lifting of* $V\beta$.

Given any functor $H : T$-**CSpa**$(\mathcal{C}) \to \mathcal{E}$ *with* $PH = V$ *and a natural transformation* $\kappa : HJ \to K$ *with* $P\kappa = 1_U$*, there is a unique natural transformation* $\overline{\kappa} : H \to \overline{K}$ *with* $P\overline{\kappa} = 1_V$ *and* $\iota \cdot \overline{\kappa}J = \kappa$.

This means: $\overline{K}$ and ι form a *right Kan extension of* K *along* J, not in **CAT**, but in the 2-category **CAT**/$\mathcal{C}$ (where 2-cells, once mapped to the level of $\mathcal{C}$, are identities).

A "universal property" in **CAT**/$\mathcal{C}$–precisely

Theorem

The functor $\overline{K} : T\text{-}\mathbf{CSpa}(\mathcal{C}) \to \mathcal{E}$ *satisfies the following properties* (a),(b),(c)*; moreover, property* (b) *determines* $\overline{K}$ *uniquely, while property* (c) *still determines* $\overline{K}$ *up to a unique natural isomorphism whose* P*-image is an identity morphism:*

(a) $P\overline{K} = V$;

(b) *there is a natural isomorphism* $\iota : \overline{K}J \to K$ *with* $P\iota = 1_U$ *and* $\iota\mathrm{B} \cdot \overline{K}\beta = \vartheta$;

(c) $\overline{K}$ *maps the pointwise* V*-cartesian transf.* β *to a pointwise* P*-cartesian lifting of* $V\beta$.

Given any functor $H : T\text{-}\mathbf{CSpa}(\mathcal{C}) \to \mathcal{E}$ *with* $PH = V$ *and a natural transformation* $\kappa : HJ \to K$ *with* $P\kappa = 1_U$*, there is a unique natural transformation* $\overline{\kappa} : H \to \overline{K}$ *with* $P\overline{\kappa} = 1_V$ *and* $\iota \cdot \overline{\kappa}J = \kappa$.

This means: $\overline{K}$ and ι form a *right Kan extension of* K *along* J, not in **CAT**, but in the 2-category **CAT**$/\mathcal{C}$ (where 2-cells, once mapped to the level of $\mathcal{C}$, are identities).

Short version of the Theorem

Corollary (Burroni 1971):

Let $P : \mathcal{E} \to \mathcal{C}$ be a fibration. Then every functor $K : T\text{-}\mathbf{ASpa}(\mathcal{C}) \to \mathcal{E}$ with $PK = U$ admits a pseudo-extension $\overline{K} : T\text{-}\mathbf{CSpa}(\mathcal{C}) \to \mathcal{E}$ over $\mathcal{C}$ (so that $\overline{K}J \cong K$ and $P\overline{K} = V$) which, up to isomorphism, is uniquely determined by the property of mapping β to a pointwise P-cartesian natural transformation.

Note:

Burroni's paper actually claims that $\overline{K}$ transforms *all* V-cartesian morphisms into P-cartesian morphisms—not just the V-cartesian morphisms β_X ($X \in T\text{-}\mathbf{CSpa}(\mathcal{C})$). We have not been able to confirm this claim and conjecture that it does not hold in general.

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The category $\mathrm{Gen}(\mathcal{C}^T)$ of T algebras with a system of generators:

$$\begin{tikzcd} (TX, \mu_X) \arrow[d, "p^\sharp"'] \\ (R, r) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ R \arrow[r, "f^*"] & S \end{tikzcd}$$

The "codomain functor" $C : \mathrm{Gen}(\mathcal{C}^T) \longrightarrow \mathcal{C}^T$ is left adjoint to the full embedding $I : \mathcal{C}^T \longrightarrow \mathrm{Gen}(\mathcal{C}^T),\ (R, r) \longmapsto (1_R, r)$, with the adjunction units $\gamma_{(p,r)} : (p, r) \longrightarrow IC(p, r)$:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "p"'] & R \arrow[d, "1_R"] \\ R \arrow[r, "1_R"] & R \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^T \arrow[rr, bend right, "I"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "U"'] & & \mathrm{Gen}(\mathcal{C}^T) \arrow[ll, bend right, "C"'] \arrow[dl, "W"] \\ & \mathcal{C} & \end{tikzcd}$$

Another topological extension of $T\text{-}ASpa(\mathcal{C})$ (T, JPAA 1979) ...

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... with the same (?) “universal property”

As a category over $\mathcal{C}$, $\mathrm{Gen}(\mathcal{C}^T)$ behaves exactly like T-**CSpa**$(\mathcal{C})$; that is:

the previous theorem remains true *verbatim*—just trade $(J, \mathrm{B}, \beta, V)$ for (I, C, γ, W):

$$
\begin{tikzcd}
T\text{-}\mathbf{ASpa}(\mathcal{C}) \simeq \mathcal{C}^T \arrow[r, "I"] \arrow[rr, bend left=40, "K"] \arrow[r, phantom, bend left=30, "\Uparrow\iota"] \arrow[dr, bend right, "U"'] & \mathrm{Gen}(\mathcal{C}^T) \arrow[r, bend left, "\overline{K}"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow\overline{\kappa}"] \arrow[r, phantom, bend left=40, "\Uparrow\kappa"] \arrow[d, "W"'] & \mathcal{E} \arrow[dl, bend left, "P"] \\
 & \mathcal{C} &
\end{tikzcd}
$$

What's going on?

Let's look at our guiding example $\mathcal{C} = \mathbf{Set}$, $T = \mathcal{U}$:

The generalities established give us the diagram (commutative in both directions)

$$\begin{tikzcd}
 & \mathcal{U}\text{-}\mathbf{ASpa} \simeq \mathbf{Set}^{\mathcal{U}} \cong \mathbf{KHaus} \arrow[dl, "J"'] \arrow[dr, "I"] & \\
\mathcal{U}\text{-}\mathbf{CAlg} \cong \mathbf{CReg} \arrow[rr, tail, "\bar{I}"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathrm{Gen}(\mathbf{KHaus}) \arrow[ll, bend left, shift left=2, "\bar{J}"]
\end{tikzcd}$$

- $\bar{I}$ maps the completely regular space X to the map $\tilde{\beta}_X : X \to \tilde{\mathrm{B}}X$, where $\tilde{\mathrm{B}}X$ is the (closed) image of the map $\beta_X^\sharp : \mathcal{U}X \to \mathrm{B}X$, $\mathfrak{x} \mapsto \lim \mathcal{U}\beta_X(\mathfrak{x})$.
- $\bar{J}$ sends an object $p : X \to R$ to the set X, provided with the weak topology with respect to p and the compact Hausdorff space R.
- $\bar{I}$ is a full embedding, NOT an equivalence of categories, with left adjoint $\bar{J}$.

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\mathcal{U}\text{-}\mathbf{CAlg} \cong \mathbf{CReg} \arrow[rr, hook, "\bar{I}"] \arrow[rr, phantom, bend right=10, "\perp"] & & \mathrm{Gen}(\mathbf{KHaus}) \arrow[ll, bend left=15, "\bar{J}"]
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A conjecture

$$
\begin{tikzcd}
T\text{-}\mathbf{ASpa}(\mathcal{C}) \arrow[r, "K"] \arrow[d, tail, "J"'] & \mathcal{E} \arrow[d, two heads, "P"] \\
T\text{-}\mathbf{CSpa}(\mathcal{C}) \arrow[r, "V"'] \arrow[ur, dashed, "\overline{K}"] & \mathcal{C}
\end{tikzcd}
$$

You guess!

"Elevated" ultrafilters

Which monads may lend themselves best for studying T-**Spa**$(\mathcal{C})$ and its subcategories?

- Already Manes (SLNM 80, 1969) studied "$(T, \widetilde{T})$-bialgebras" over **Set**, under some conditions on the interaction of the two participating monads, considering in particular the case $T = M \times (\text{-})$ for a monoid M and $\widetilde{T} = \mathcal{U}$.
- Leinster (TAC 2013) showed the importance of looking at $\mathcal{U}$ as a codensity monad (originally due to Kennison and Gildenhuys), and gave other significant examples in this context, like the double-dualization monad for vector spaces.
- Adámek and Sousa (Adv. Math. 2020) introduced the notion of "D-ultrafilter" (with D referring to a "dualizing object" and cogenerator in its category, a very powerful notion that captures in particular the examples already mentioned and deepens their study.

Let us look at one of their examples, without alluding to its codensity status!

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The simplified algebraic theory of M-sets

Let $M = (M, e, m)$ be a monoid.

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\begin{aligned}
X \text{ is an } M\text{-set} \quad &\iff X \text{ comes with a unital and associative action } M \times X \to X \\
&\iff X \text{ comes with a homomorphism } M \to \mathbf{Set}(X, X) \text{ of monoids} \\
&\iff X \textit{ is } \text{a functor } M \to \mathbf{Set}
\end{aligned}
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Lawvere and Eilenberg-Moore in total harmony: $\mathbf{Set}^M = [M, \mathbf{Set}]$

Actually, **Set** may be replaced by *any* category $\mathcal{C}$; define $\mathcal{C}^M := [M, \mathcal{C}]$.

For example: $\mathbf{Top}^M$ is the category of M-sets that come equipped with a topology which makes all translation maps $s : X \to X$ $(s \in M)$ continuous;
morphisms are continuous equivariant maps.

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The ultrafilter monad of $\mathbf{Set}^M$

Now, let M be *commutative*
(so that the translation maps of any M-set become morphisms of $\mathbf{Set}^M$),

and let X be an M-set. Then:

- Make $\mathcal{P}X$ an M-set by $sA = s^{-1}(A) = \{x \in X \mid sx \in A\}$ $(A \subseteq X)$; this lifts the contravariant powerset functor from $\mathbf{Set}$ to $\mathbf{Set}^M$.
- $\mathcal{U}$, as a subfunctor of $\mathcal{P}\mathcal{P}$, lifts from $\mathbf{Set}$ to $\mathbf{Set}^M$ as well; so, $\mathcal{U}X$ is an M-set via $s\mathfrak{x} = \{A \subseteq X \mid sA \in \mathfrak{x}\}$, for all $\mathfrak{x} \in \mathcal{U}X$.
- The monad unit and multiplication of $\mathcal{U}$ become componentwise equivariant; so, we have the lifting $\mathcal{U}^M$ of the monad $\mathcal{U}$ from $\mathbf{Set}$ to $\mathbf{Set}^M$.
- The Eilenberg-Moore category of $\mathcal{U}^M$ is $\mathbf{KHaus}^M$ and, not surprisingly, one has
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Fortunately, also Stone-Čech lifts to $\mathbf{Set}^M$!

For an M-set X, make $\mathrm{B}X$ an M-set, such that $\beta_X : X \to \mathrm{B}X$ becomes eqivariant, by:

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Proof that this works and gives the desired adjunction: **CAT** is a 2-category!

$$\begin{tikzcd} {[M, \mathbf{KHaus}]} \arrow[r, bend right, "{[M,\mathrm{Inc}]}"'] \arrow[r, phantom, "\perp"] & {[M, \mathbf{Top}]} \arrow[l, bend right, "{[M,\mathrm{B}]}"'] \end{tikzcd}.$$

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More examples?

Wait and see!

THANK YOU!

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