Limits and colimits in synthetic ∞-categories

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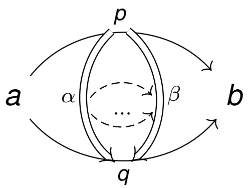
∞-categories

- Idea of an ∞-category: It is what follows in 1-category, 2-category, ..., n-category ...
- More precisely: $(\infty, 1)$-category.



- Problem: Many definitions, and none of them is easy.
- Possible solution: Synthetic theory (Riehl-Shulman, Riehl-Verity).
- Our contribution: Theory of limits and colimits.

Outline

Homotopy type theory

Via the **Curry-Howard-Voevodsky**[1] correspondence:

Set theory	**Logic**	**Homotopy theory**	**HoTT**
sets	propositions	spaces	Types $A, B, \ldots$
elements	proofs	points	$a : A$
$\emptyset, \{\emptyset\}$	$\top, \bot$	$\emptyset, \star$	$0, 1, \mathcal{U}$
product	conjunction	product	$A \times B$
disjoint union	disjunction	coproduct	$A + B$
functions	implication	function space	$A \to B$
family of sets	predicate	fibration	$B : A \to \mathcal{U}$
product	$\forall$	space of sections	$\prod_{a:A} B(a)$
disjoint union	$\exists$	total space	$\sum_{a:A} B(a)$

[1] Emily Riehl

Identity type

- Formation: For A Type and $a, b : A$ we form $a =_A b$ Type.
- Introduction: For all $a : A$ there is $\mathrm{refl}_a : a =_A a$.
- Elimination*: For every family type $C : A \to A \to \mathcal{U}$ if

$$u : \prod_{a:A} C(a, a)$$

then

$$f : \prod_{a,b:A} \prod_{p:a=b} C(a, b).$$

- Computation*: $f(a, a, refl_a) \equiv u(a)$.

Identity type

- Elimination: For every family type

$$C : \prod_{a,b:A} (a = b) \to \mathcal{U}$$

  and

$$u : \prod_{a:A} C(a, a, \mathrm{refl}_a)$$

  there is

$$f : \prod_{a,b:A} \prod_{p:a=b} C(a, b, p)$$

Path ind.

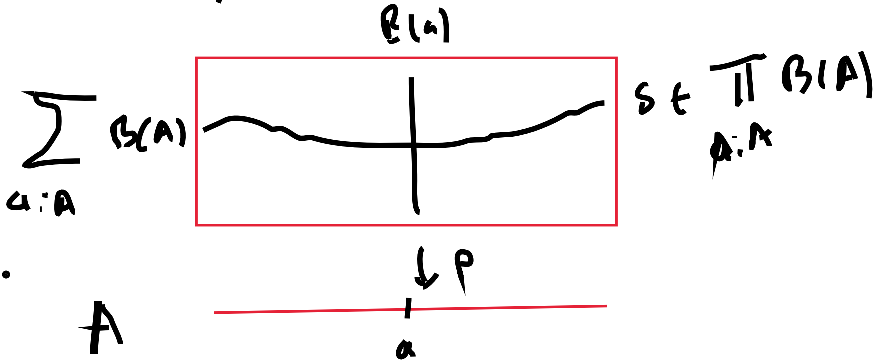
- Computation: $f(a, a, \mathrm{refl}_a) \equiv u(a)$

A model for HoTT

Theorem (Voevodsky-Kapulkin-Lumsdaine)

The category $\mathbf{Set}^{\Delta^{op}}$ *supports an interpretation of HoTT where types correspond to Kan complexes.*

The nicest possible situation we can draw:



Simplicial HoTT

Can be obtained by adding a strict interval to HoTT:
A type $\mathbb{2}$ with distinct endpoints $0, 1$ with an order relation, plus some axioms making it into an interval.

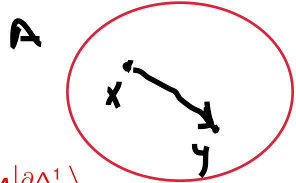
Theorem (Riehl-Shulman)

The category $\mathbf{sSet}^{\Delta^{op}}$ *supports an interpretation of sHoTT where Segal types correspond to Segal spaces and Rezk types to Rezk spaces.*

Rezk spaces are certain bisimplicial sets which are our $(\infty, 1)$-categories in this model.

We can construct some "simplicial sets".

Simplicial HoTT



Furthermore:

- $$\hom_A(x,y) := \left\langle \Delta^1 \to A\Big|^{\partial\Delta^1}_{[x,y]} \right\rangle.$$

  If $f : \hom(x,y)$ then $f(0) \equiv x$ and $f(1) \equiv y$.
- For $x : A$ define $\mathrm{id}_x := \lambda(t : \Delta^1).x$.
- The type of compositions for $f : \hom_A(x,y)$ and $g : \hom_A(y,z)$:

$$\hom^2_A\left(\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & \xrightarrow[h]{} & z \end{array} \right) := \left\langle \Delta^2 \to A\Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle.$$

Some types

- A type A is **contractible** if

$$\sum_{a:A} \prod_{x:A} a = x.$$

- A type A is a **proposition** if

$$\prod_{x,y:A} x = y.$$

Segal types

Segal types are types with "unique composition". By definition, if the type

$$\sum_{\underline{h:\hom_A(x,z)}} \hom^2_A \left(\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & \xrightarrow[h]{} & z \end{array} \right)$$

is contractible.

- This is enough to get the categorical structure.
- In Segal types we have associative and unital composition, denoted $g \circ f$.
- Functions $f : A \to B$ between Segal types are "functors".

Rezk types

- An arrow $f : \hom_A(x, y)$ is an **isomorphism** if

$$\left(\sum_{g:\hom_A(y,x)} g \circ f = \mathrm{id}_x\right) \times \left(\sum_{h:\hom_A(y,x)} f \circ h = \mathrm{id}_y\right).$$

- The type above is a proposition so:

$$(x \cong y) := \sum_{f:\hom_A(x,y)} \mathrm{isiso}(f).$$

- By path induction

$$\mathrm{idtoiso} : \prod_{x,y:A} (x = y) \to (x \cong y).$$

  in the case $x = x$ define $\mathrm{idtoiso}(\mathrm{refl}_x) := \mathrm{id}_x$.

A Segal type is **Rezk** if idtoiso is an equivalence.

Natural transformations and adjunctions

A **natural transformation** is an element $\alpha : \hom_{A \to B}(f, g)$.

- Component-wise determined

$$\hom_{A \to B}(f, g) \simeq \prod_{a:A} \hom_B(f(a), g(a)).$$

- The type theory makes them natural.

A **quasi-transposing adjunction** between types A, B consist of **functors** $f : A \to B$ and $u : B \to A$ and a family of equivalences

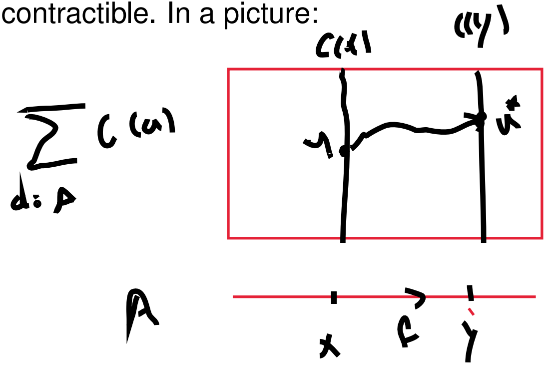
$$\phi : \prod_{a:A, b:B} \hom_B(fa, b) \simeq \hom_A(a, ub).$$

Covariant families

Covariant families are the analogous to Grothendieck fibration.
A family $C : A \to \mathcal{U}$ is **covariant** if for each $x, y : A$, $f : \hom_A(x, y)$ and $u : C(x)$ the type

$$\sum_{v : C(y)} \hom_{C(f)}(u, v)$$

is contractible. In a picture:



Closure properties

Theorem (Riehl-Shulman)

If $B : A \to \mathcal{U}$ *is a covariant family type over* A *a Segal type such that for all* $a : A$ *the type* $B(a)$ *is a Segal type then* $\prod_{a:A} B(a)$ *and* $\sum_{a:A} B(a)$ *are Segal type. In particular* $A \to B$ *is Segal if* B *is a Segal type.*

Theorem (B.)

If A *is a Rezk type and* $C : A \to \mathcal{U}$ *is a covariant family over* A *then* $\sum_{x:A} C(x)$ *is also a Rezk type.*

Initial/terminal and (co)cones

An element $b : B$ is **initial** if for all $x : B$, $\hom_B(b, x)$ is contractible.

$$\mathrm{isinitial}(b) := \prod_{x:B} \mathrm{iscontr}(\hom_B(b, x))$$

Also, an element $b : B$ is **terminal** if for all $x : B$, $\hom_B(x, b)$ is contractible.

$$\mathrm{isterminal}(b) := \prod_{x:B} \mathrm{iscontr}(\hom_B(x, b)).$$

Define the type of **(co)cones** of $f : A \to B$

$$\mathrm{cocone}(f) := \sum_{b:B} \hom_{A \to B}(f, \triangle b),$$

$$\mathrm{cone}(f) := \sum_{b:B} \hom_{A \to B}(\triangle b, f).$$

Synthetic (co)limits

Definition

*A **colimit** for $f : A \to B$ is an initial element of the type:*

$$\mathrm{cocone}(f) := \sum_{x:B} \hom_{A \to B}(f, \triangle x).$$

*A **limit** for $f : A \to B$ is an terminal element of the type:*

$$\mathrm{cone}(f) := \sum_{x:B} \hom_{A \to B}(\triangle x, f).$$

Theorem (B.)

Under interpretation, the definition above coincides with the one given in bisimplicial sets.

Classical results

Theorem (B.)

1. *Colimits are unique up to isomorphism if they exist.*
2. *There exist a colimit* (a, α) *for* f *if and only if*

$$\prod_{x:B}(\mathrm{hom}_B(a, x) \simeq \mathrm{hom}_{A\to B}(f, \triangle x)).$$

Proof.

1. Prove initial elements are unique up to isomorphism.
2. Use Yoneda lemma.

□

RAPL/LAPC

In the presence of an adjunction then we also have the usual preservation of limits and colimits.

Theorem (B)

Let A, B be Segal types and functions $g : J \to B$, $f : A \to B$, $u : B \to A$, such that g has a limit (b, β) and u is right quasi-transposing adjunction of f then $(u(b), u\beta)$ is a limit for $ug : J \to A$.

Proof.

It is the same as in categories modulo some type theory complications. □

Special case: Rezk types

Let B is a Segal type a $f : A \to B$ a function, we define the type:

$$\text{colimit}(f) := \sum_{w:\text{cocone}(f)} \text{isinitial}(w),$$

and also

$$\text{limit}(f) := \sum_{w:\text{cone}(f)} \text{isterminal}(w).$$

Lemma

The types colimit(f) *and* limit(f) *are propositions.*

Proof.

- Use that (co)limits are unique up to isomorphism. Any two (co)limits $(a, \alpha), (b, \beta)$ are isomorphic $(a, \alpha) \cong (b, \beta)$
- The types above are Rezk types hence $(a, \alpha) \cong (b, \beta)$ is equivalent to $(a, \alpha) = (b, \beta)$.

□

Corollary (B.)

In Rezk types, (Co)limits are unique up to equality.

Limit of spaces: Motivation

Let $\{G_i\}_{i \in I}$ be a family of ∞-groupoids indexed by a set I. Denote by G to the obvious diagram $I \to \infty$-**Gpd** then we have that

$$lim_I G_i = \prod_{i \in I} G_i.$$

Problem:There is not something like an ∞-category of ∞-groupoids

Solution: *univalent covariant families* due to Cavallo-Riehl-Sattler.

Covariant univalent families

Fix a covariant family $E : B \to \mathcal{U}$ over a Segal type B and $a, b : B$ we obtain a function $f \mapsto (u \mapsto u^{*})$

$$\text{arrtofun} : \hom_B(a, b) \to (E(a) \to E(b)).$$

E is **covariant univalent** if for all $a, b : B$ the map arrtofun is an equivalence.

For a type A define

$$\text{issmall}_B(A) := \sum_{b:B} (E(b) \simeq A)$$

A is B-**small** if $\text{issmall}_B(A)$.

The type B is regarded as a "∞-category" of "∞-categories", it is definable in sHoTT and consistent (C-R-S).

Limit as dependant product

Let B a Rezk type, a function $f : D \to B$ and assume that $E : B \to \mathcal{U}$ is a univalent covariant family

Proposition (B.)

If (b_0, σ_0) is the center of contraction of $\mathrm{issmall}_B(\prod_{d:D} E(f(d)))$ then b_0 is the limit for f.

$E(b_0)$

Proposition (B.)

If $f : D \to B$ has limit (b_0, α) and there is $b_1 : B$ and $u : E(b_1)$. Then $\prod_{d:D} E(f(d))$ is B-small.