Leinster's Generalized Mobius Inversion

Classical Mobius Inversion

Let α, β be sequences of complex #s

$$\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots$$

$$\beta = \beta_1, \beta_2, \beta_3, \ldots$$

These sequences have a convolution product

$$(\alpha * \beta)_n = \sum_{km=n} \alpha_k \beta_m$$

Key points:

1) $*$ is an associative operation. In particular is well-defined, since the sum is finite.

2) There is a unit:

$$e = 1, 0, 0, 0, 0, \ldots$$

Let $\zeta = 1, 1, 1, 1, 1, \ldots$

Thm: $\mathcal{I}$ has an inverse under convolution, called the mobius function

$$\mu_n = \mu(n) = \begin{cases} +1 & \text{if } n \text{ is square free \& has an even \# of prime factors} \\ -1 & \text{if square free and an odd \# of prime factors} \\ 0 & \text{if } n \text{ is divisible by a square} \end{cases}$$

Also
$\mu(1) = 1$

Lots of fantastic facts about μ.

1) μ is <u>multiplicative</u>, i.e.

$\mu(mn) = \mu(m)\mu(n)$ if m, n are relatively prime.

2) Let f be any arithmetic function, i.e. $f: \mathbb{N} \to \mathbb{C}$. Define a second arithmetic function by

$$g(n) = \sum_{d \mid n} f(d)$$

Then

$$f(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right)$$

3) Consider the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad Re(s) > 1$$

$$\zeta(s) = \frac{1}{\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}}$$

There are multiple Wikipedia articles on these topics. GC Rota worked on generalizing these ideas to more general posets. The original setting is all about the poset of natural #'s ordered under divisibility. The key is always to make sure the sum is finite. See Stanley

- Enumerative Combinatorics

<u>Next Step</u>: Generalize from posets to categories. Which categories allow for this type of sum to be finite

Initial Work by Leroux, Haigh, etc

First observation is that $\mathbb{C}$ can be replaced with more general rings or rigs.

Def: A category $\mathcal{C}$ is <u>finely finite</u> if $\forall$ arrows $f: A \to B$, there are only finitely many commutative diagrams of the form

$$\begin{tikzcd} & C \arrow[dr, "h"] & \\ A \arrow[ur, "g"] \arrow[rr, "f"'] & & B \end{tikzcd} \qquad f = h \circ g$$

Consider functions

$$\beta, \alpha : Ar(\mathcal{C}) \to R$$

where R is a rig.

Define $\alpha * \beta : Ar(\mathcal{C}) \to R$

$$\alpha * \beta(f) = \sum_{hg = f} \alpha(g)\beta(h)$$

denoted $k\mathcal{C}$

This is called the <u>fine incidence algebra</u>; $k\mathcal{C}$

We have the function $\zeta_{\mathcal{C}} : Ar(\mathcal{C}) \to k$

$$\zeta_{\mathcal{C}}(f) = 1 \quad \forall f \in Ar(\mathcal{C})$$

$\mathcal{C}$ has <u>Mobius inversion over k</u> if

ζ is invertible. $\mu_{\mathcal{C}} = \zeta_{\mathcal{C}}^{-1}$ is called the fine Mobius function.

Thm: Let $\mathcal{C}$ be a poset viewed as a category. $\mathcal{C}$ is finely finite iff it is locally finite, i.e. $\forall a, b$ the set $\{c \mid a \leq c \leq b\}$ is finite.

Then $Ar(\mathcal{C}) = \{(a,b) \mid a \leq b\}$

The mobius function, if it exists, satisfies

$$\sum_{c \mid a \leq c \leq b} \mu(a,c) = \sum_{c \mid a \leq c \leq b} \mu(c,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if not} \end{cases}$$

If k is a ring, the Mobius function exists

$$M(a,b) = \sum_{n \in \mathbb{N}} (-1)^n \cdot \#\text{ of chains of length } n \quad a = a_0 < \cdots < a_n = b$$

This is due to P. Hall (1936)

Now consider finite categories, i.e. objects and arrows are finite, and let k be a rig.

The <u>coarse incidence algebra</u> $k_c A$ is the set of all functions

$$Ob(A) \times Ob(A) \to k$$

$\alpha, \beta \in k_c A$, define

$$(\alpha * \beta)(a,b) = \sum_{c \in A} \alpha(a,c)\beta(c,b)$$

The analogue of the ζ sequence is the coarse ζ function

$$\zeta_C(a,b) = |\mathrm{Hom}(a,b)|$$

the cardinality of the Hom Set. Say that $\mathcal{C}$ has coarse Mobius inversion over R if ζ has an inverse, M_C.

Thm (Haigh) Let $\mathcal{C}$ be a finite category. If $\mathcal{C}$ has fine mobius inversion over R, then it also has coarse Mobius inversion over R, and the formula can be written explicitly:

$$M_C(a,b) = \sum_{f: A \to B} M(f)$$

Proof: Straight calculation or see below

Functoriality: What do functors do on the corresponding algebras? In general, not well-defined

Let $\mathcal{C}, \mathcal{D}$ be finely finite categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with finite fibres,

i.e. $\forall g \in Ar(\mathcal{D})$ $F^{-1}(g)$ is a finite set.

This is sufficient to imply there is a

k-linear map

$$F_! : k\mathcal{C} \to k\mathcal{D}$$

If $\alpha \in kA$, define

$$F_!(\alpha)(g) = \sum_{\{f \mid Ff = g\}} \alpha(f)$$

The fact that fibers are finite implies this is well-defined and k-linear. But not necessarily an algebra map.

Thm (Content - Lemmay & Leroux)

Let $\mathcal{C}, \mathcal{D}$ be finely finite, $F: \mathcal{C} \to \mathcal{D}$ a functor with finite fibres. Then $F_! : k\mathcal{C} \to k\mathcal{D}$ is an algebra homomorphism for all rings k iff F is bijective on objects.

<u>EX1</u>: If $\mathcal{C}$ is a cat, let $C\mathcal{C}$ be the corresponding codiscrete category. Let $\mathcal{C}$ be finite. Evidently,

$$\underbrace{k_c \mathcal{C}}_{\text{coarse incidence algebra}} \cong \underbrace{k(C\mathcal{C})}_{\text{fine incidence algebra of codiscrete category}}$$

There is a canonical functor $\mathcal{C} \to C\mathcal{C}$. This induces an algebra map

$$\Sigma : k\mathcal{C} \to k_c(\mathcal{C})$$

defined by

$$\Sigma \alpha(a,b) = \sum_{f: a \to b} \alpha(f)$$

The image under Σ of the fine zeta function is the coarse zeta function.
This can be used to prove Haigh's construction

Let $\mathcal{C}$ be a category. Let a, b be objects of $\mathcal{C}$. The patch $[a,b]$ is the full subcategory consisting of objects c s.t. $\exists$ maps

$$a \to c \to b$$

Lemma: TFAE

1) $\forall\, a, b \in Ob(\mathcal{C})$, the patch $[a,b]$ is finite
2) $\mathcal{C}$ is finely finite and has finite homsets
3) $\forall\, a, b \in Ob(\mathcal{C})$, the set

$$\{c \in Ob(\mathcal{C}) \mid \exists\, a \to c \to b\} \text{ is finite}$$

& $\mathcal{C}$ has finite homsets

Such a category is called <u>patch-finite</u>

(coarsely) finite $\Rightarrow$ patch finite $\Rightarrow$ finely finite

Patch-finite categories are "finite enough" to do coarse Mobius inversion

Let $P(\mathcal{C})$ be the preorder associated to $\mathcal{C}$,

i.e. $a \le b$ iff $\exists f: a \to b$

Then define $k_p(\mathcal{C}) = k(P(\mathcal{C}))$

So $k_p(\mathcal{C}) = \{(a,b) \mid \mathrm{Hom}(a,b) \neq \emptyset\} \to k$

or equivalently

$$k_p \mathcal{C} = \{\alpha \in k_c \mathcal{C} \mid \mathrm{Hom}(a,b) = \emptyset \Rightarrow \alpha(a,b) = 0\}$$

$$\subseteq k_c \mathcal{C} \text{ (as a } k\text{-space)}$$

$k_c(\mathcal{C})$ is not an algebra

But for the subspace $k_p(\mathcal{C})$ * the multiplication is well-defined

But the multiplication is well-defined ~~B~~ on

$\Bbbk_p \mathcal{C}$

$$(\alpha * \beta)(a,b) = \sum_{c \in [a,b]} \alpha(a,c)\beta(c,b)$$

Patch finiteness ensures this is finite.

I have a functor

$$\mathcal{C} \to P(\mathcal{C})$$

which is bijective on objects and has finite fibres

So I have a homomorphism

$$\Sigma : \Bbbk\mathcal{C} \to \Bbbk P(\mathcal{C})$$

All of the above can be redone on ~~the~~ patch-finite categories via the homomorphism Σ.

Möbius Inversion for enriched categories

Reminder: Suppose V is a monoidal category

A V-category consists of the following. A class of objects, an assignment $V(a,b) \in |V|$ for every pair of objects, a composition operation, i.e.

an arrow $$V(b,c) \otimes V(a,b) \to V(a,c)$$

an identity arrow for all objects a

$$I \to V(a,a)$$

satisfying appropriate axioms.

Ex, $Vec_{\mathbb{R}}$, Ban, etc

Lawvere's ex Let $V = ([0,\infty], \geq)$ opposite of usual order

$\otimes$ is addition.

Let X be any set viewed as a set of objects

$a,b \in X$ $V(a,b) \in [0,\infty]$

composition says

$$V(a,b) + V(b,c) \geq V(a,c)$$

Lawvere called these V-categories *generalized metric spaces*. Note we also have

$$V(a,a) \leq 0, \text{ so } V(a,a) = 0.$$

But we don't have symmetry or

$$V(a,b) = 0 \Rightarrow a = b.$$

To define V-mobius inversion, we need a map

$$|-| : V_0/_{\cong} \to (\mathbb{R}, \cdot, 1)$$

s.t. $\qquad |a \otimes b| = |a||b|$

Suppose $\mathcal{C}$ is a finite V-category. The *coarse incidence algebra* is defined as in non-enriched case. Consider

$$\alpha : \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \mathbb{R}.$$

The coarse zeta function is

$$\zeta(a,b) = |V(a,b)|$$

Take $k = \mathbb{R}$ and $|\ | : [0,\infty] \to \mathbb{R}$ to be

$$|x| = e^{-x}$$

This gives a notion of mobius inversion for metric spaces. Most (but not all) finite metric spaces have mobius inversion. If it exists, define the <u>magnitude of the space</u> to be

$$\sum_{a,b} \mu(a,b)$$

See Leinster: The magnitude of a metric space