Divided power algebras

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Classical divided power algebras

Classifying space

For G a group, a classifying space is a topological space BG such that $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for all $n \neq 1$.

Fact 1 : There always is a BG, unique up to homotopy.

Fact 2 : There is a natural construction $G \mapsto BG$ using the Bar construction. In particular, BG can be seen as a simplicial set.

Classifying space

Let $\mathbb{F}$ be a field. $H_*(BG, \mathbb{F})$ is the homology of G with coef. in $\mathbb{F}$.

The group structure of G induces a multiplication :

$$* : BG \times BG \cong B(G \times G) \to BG$$

This induces, through the Eilenberg–Zilber map, a multiplication :

$$\begin{array}{rccc} * : & H_i(BG, \mathbb{F}) \otimes H_j(BG, \mathbb{F}) & \to & H_{i+j}(BG, \mathbb{F}). \\ & ([x], [y]) & \mapsto & \sum_{(\mu,\nu) \in Sh(i,j)} [(s_\nu x) * (s_\mu y)] \end{array}$$

where :
$Sh(i, j) = \{$ partitions (μ, ν) with $(\mu_1 < \cdots < \mu_i), (\nu_1 < \cdots < \nu_j)\}$

$$s_\mu = (s_{\mu_1 - 1} \dots s_{\mu_i - 1}) \qquad s_\nu = (s_{\nu_1 - 1} \dots s_{\nu_j - 1}).$$

First example : $H_*(\mathbb{R}P^\infty, \mathbb{F}_2)$

Let $\mathbb{F} = \mathbb{F}_2$, $G = \mathbb{Z}/2\mathbb{Z}$.

Fact 1 : The infinite real projective space $\mathbb{R}P^\infty$ is a $B(\mathbb{Z}/2\mathbb{Z})$.

Fact 2 : $\bigoplus_n H_n(\mathbb{R}P^\infty, \mathbb{F}) \cong \mathbb{F}[x]$ as $\mathbb{F}$-vector spaces.

Notation : $x^{[n]} \longleftrightarrow x^n$

Fact 3 : The multiplication $*$ is induced by :

$$x^{[m]} * x^{[n]} = \binom{m+n}{m} x^{[m+n]}$$

Denote by $\Gamma(x) = (\bigoplus_n H_n(\mathbb{R}P^\infty, \mathbb{F}), *)$.

Remark : As an algebra, $\mathbb{F}[x]$ is generated by x. $\Gamma(x)$ is not :

$$x * x = 2\, x^{[2]} = 0.$$

However, $\Gamma(x)$ is generated by $\{x^{[2^k]}\}_{k \in \mathbb{N}}$.

More examples

Over any $\mathbb{F}$, I can define the algebra $(\Gamma(x), *)$ as above.

If $char(\mathbb{F}) = 0$, $\Gamma(x)$ is isomorphic to $\mathbb{F}[x]$ as an algebra :

$$f : \Gamma(x) \to \mathbb{F}[x], \qquad x^{[n]} \mapsto \frac{x^n}{n!}.$$

If $char(\mathbb{F}) = p > 0$, $\Gamma(x)$ is generated by $\{x, x^{[p]}, x^{[p^2]}, \ldots\}$.

Define $\gamma_n(x) = x^{[n]}$.

Definition (Cartan 1954)

A divided power algebra is a commutative associative unital algebra A with a family of operations (set-maps) $\gamma_n : \bar{A} \to A$ satisfying :

(1) $\gamma_o(x) = 1$, $\gamma_1(x) = x$, $\deg \gamma_k(x) = k.\deg(x)$.

(2) $\gamma_k(x)\ \gamma_h(x) = (k,h)\ \gamma_{k+h}(x)$, en notant (k,h) le coefficient binominal $\frac{(k+h)!}{k!h!}$

(3) $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\ \gamma_j(y)$ (formule de Leibniz)

(4) $\gamma_k(xy) = k!\ \gamma_k(x)\ \gamma_k(y) = x^k\ \gamma_k(y) = \gamma_k(x)y^k$

(5) $\gamma_k(\gamma_h(x)) = (h,h-1)(2h,h-1)\ldots((k-1)h,h-1)\ \gamma_{kh}(x)$.

[HC] Henri Cartan, Puissance Divisées, Séminaire Henri Cartan de l'ENS, 1954/1955, Algèbre d'Eilenberg MacLane et homotopie.

Definition (Cartan 1954)

A divided power algebra is a commutative associative unital algebra A with a family of operations (set-maps) $\gamma_n : \bar{A} \to A$ satisfying :

(H) $\gamma_0(x) = 1$, (U) $\gamma_1(x) = x$, (M) $\deg \gamma_k(x) = k.\deg(x)$.

(R/R) $\gamma_k(x)\ \gamma_h(x) = (k,h)\ \gamma_{k+h}(x)$, en notant (k,h) le coefficient binominal $\dfrac{(k+h)!}{k!h!}$

(L) $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\ \gamma_j(y)$ (formule de Leibniz)

(C) $\begin{cases} \gamma_k(xy) = k!\ \gamma_k(x)\ \gamma_k(y) = x^k\ \gamma_k(y) = \gamma_k(x) y^k \\ \gamma_k(\gamma_h(x)) = (h,h-1)(2h,h-1)\ldots((k-1)h,h-1)\ \gamma_{kh}(x) . \end{cases}$

1) S : Symmetry (Commutativity of the multiplication)
2) H : Homogeneity,
3) M : Monomiality $\rightsquigarrow \gamma_k(\lambda x) = \lambda^k \gamma_k(x)$,
4) R/R : Reduction with respect to the repetition of inputs,
5) L : Leibniz formula,
6) U : Unitality,
7) C : Composition.

Founding results

Theorem (Roby 1965)

For V an $\mathbb{F}$-vector space, the free divided power algebra $\Gamma(V)$ generated by V is given by $\Gamma(V) = \bigoplus_n (V^{\otimes n})^{\mathfrak{S}_n}$.

$$\underline{v} * \underline{w} = \sum_{\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m} \sigma \cdot (\underline{v} \otimes \underline{w}),$$

Sh(n,m) .

and $\gamma_n(\underline{v}) = \underline{v}^{\otimes n}$.

Remark : $\Gamma(x) = \Gamma(\mathbb{F})$.

Theorem (Cartan 1955)

If A is a simplicial commutative, associative algebra, then its homotopy $\pi(A)$ is a divided power algebra.

Remark : This implies that $\bigoplus_n H_n(BG, \mathbb{F})$ is a divided power algebra for G abelian.

Modern version

From now on, $\Gamma(V) = \bigoplus_{n>0} (V^{\otimes n})^{\mathfrak{S}_n}$.

Γ has a *monad* structure. $\Gamma(V)$ is spanned by "monomials"

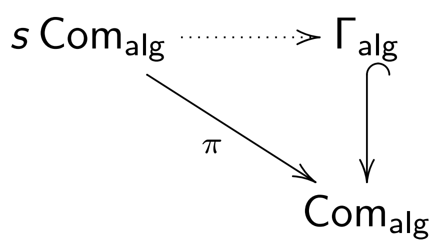
$$\mathfrak{t} = \gamma_{r_1}(v_1) * \cdots * \gamma_{r_s}(v_s),$$

The *unit* $V \to \Gamma(V)$ is $v \mapsto \gamma_1(v)$, and the *composition* $\Gamma(\Gamma(V)) \to \Gamma(V)$ is given on monomials

$$\gamma_{q_1}(\mathfrak{t}_1) * \cdots * \gamma_{q_{s'}}(\mathfrak{t}_{s'})$$

by the relations of divided power algebras (L), (M), (C). . .

<u>Cartan's result</u> :



Digression : Deriving divided power polynomials

Instinctively, $\left(\frac{x^n}{n!}\right)' = \frac{x^{n-1}}{(n-1)!}$, so $\gamma_n' = \gamma_{n-1}$.

Define $d_V : \Gamma(V) \to \Gamma(V \times V)$ by :

$$d_V(\gamma_{r_1}(v_1) * \cdots * \gamma_{r_s}(v_s)) = \sum_{i=1}^{s} \gamma_{r_1}(v_1, 0) * \cdots * \gamma_{r_i - 1}(v_i, 0) * \cdots * \gamma_{r_s}(v_s, 0) * (0, v_i)$$

$$\text{“} = \sum_{i=1}^{n} \gamma_{r_1}(v_1) * \cdots * \gamma_{r_i - 1}(v_i) * \cdots * \gamma_{r_s}(v_s) * dv_i \text{”}$$

Digression : Deriving divided power polynomials

The Kleisli category of Γ has objects vector space and morphisms

$$f : V \rightsquigarrow W$$

are linear maps

$$[f] : V \to \Gamma(W).$$

$f : V \rightsquigarrow W$ has "total derivative" a map $Df : V \rightsquigarrow W \times W$,

$$[Df] : V \xrightarrow{[f]} \Gamma(W) \xrightarrow{d_W} \Gamma(W \times W)$$

In the opposite of the Kleisli category, we obtain a derivative of maps, in the sense of Blute–Cockett–Seely.

[BCS] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely, Differential categories (2006).

Definition : A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear operation $[-;-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying :

$$[a;b] = -[b;a], \qquad [a;[b;c]] + [c;[a;b]] + [b;[c;a]] = 0.$$

In characteristic p, a p-restricted Lie algebra is a Lie algebra endowed with an operation $a \mapsto a^{[p]}$ such that :

$$\begin{aligned}
&(M) & (\lambda a)^{[p]} &= \lambda^p a^{[p]}, \\
&(L) & (a+b)^{[p]} &= a^{[p]} + b^{[p]} + \textstyle\sum_{i=1}^{p-1} s_i(a,b), \\
&(C) & [a^{[p]};b] &= \underbrace{[a;[a;\dots[a;}_{p}b]]\dots],
\end{aligned}$$

where $s_i(a,b)$ are Lie polynomials in a and b.

Examples of Restricted Lie algebra

Over a field of characteristic p,

Example 1 : If A is an associative algebra, define a Lie bracket on A by $[a; b] = ab - ba$. Then the operation $a \mapsto a^p$ makes A into a p-restricted algebra.

Example 2 : The algebra of derivations of a (non-necessarily associative) algebra is a restricted Lie algebra for the p-th power...

Example 3 : The Lie algebra associated to an algebraic group over a field of characteristic p is a p-restricted Lie algebra (A. Borel, 91).

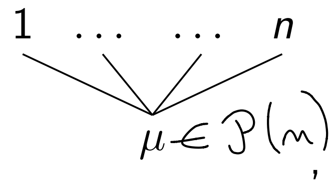
Example 4 : The homotopy of a simplicial Lie algebra is a p-restricted Lie algebra.

Operads

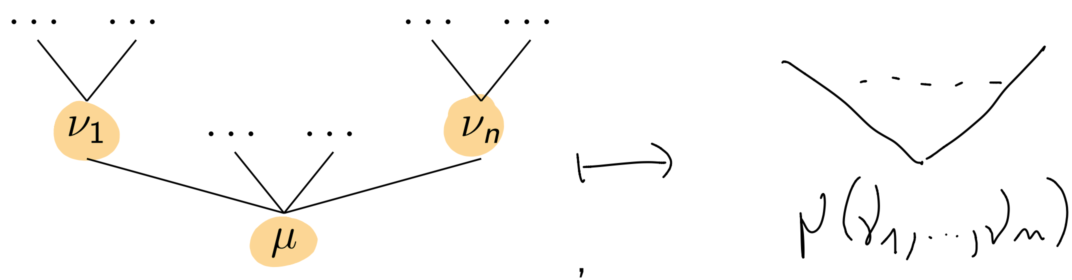
Definition

"An operad encodes operations"

An operad $\mathscr{P} = (\mathscr{P}(n))_n$ is a sequence of vector spaces $\mathscr{P}(n)$ of "arity n operations" :



together with actions $\mathfrak{S}_n \curvearrowright \mathscr{P}(n)$ *"permuting inputs"*,
Composition of operations :



A unit for composition $1_{\mathscr{P}} \in \mathscr{P}(1)$,
$+$ compatibility relations (unitality, associativity, equivariance).

Operads as monoids

A symmetric sequence is a sequence $(\mathscr{M}(n))_n$ of $\mathfrak{S}_n$-representations.

The tensor product of symmetric sequences is given by :

$$(\mathscr{M} \otimes \mathscr{N})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}} \mathscr{M}(i) \otimes \mathscr{N}(j).$$

The composition product of symmetric sequences is given by :

$$(\mathscr{M} \circ \mathscr{N})(n) = \bigoplus_{k \geq 0} \mathscr{M}(k) \otimes_{\mathfrak{S}_k} (\mathscr{N}^{\otimes k})(n).$$

An operad is a monoid for $\circ$.

Algebras

An operad has "algebras".
$\mathscr{P}$-algebra = vector space endowed with the operations of $\mathscr{P}$.

Examples : As, Com, Lie . . . $\mathrm{Com}(n) = \mathbb{F}$ trivial rep. for $n > 0$.

More precisely : For each operad $\mathscr{P}$, there is a monad $S(\mathscr{P}, -)$ in vector space :

$$S(\mathscr{P}, V) = \bigoplus_{n \geq 0} \mathscr{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n} = \mathscr{P} \circ V.$$

$V \to \mathscr{P} \circ V$.

$\mathscr{P} \circ \mathscr{P} \circ V \to \mathscr{P} \circ V$.

$V = V(0)$,

Examples :

$S(\mathrm{As}, -) =$ tensor algebra,
$S(\mathrm{Com}, -) =$ symmetric algebra,
$S(\mathrm{Lie}, -) =$ free Lie algebra. . .

Definition : a $\mathscr{P}$-algebra is an algebra over the monad $S(\mathscr{P}, -)$.

The Γ functors

Define another composition product on symmetric sequences :

$$(\mathscr{M} \tilde{\circ} \mathscr{N})(n) = \bigoplus_{k \geq 0} \left(\mathscr{M}(k) \otimes (\mathscr{N}^{\otimes k}(n))\right)^{\mathfrak{S}_k}.$$

Define a functor $\Gamma(\mathscr{P}, -)$ such that :

$$\Gamma(\mathscr{P}, V) = \bigoplus_{n \geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{\mathfrak{S}_n} = \mathscr{P} \tilde{\circ} V$$

Remark : For $\mathscr{P} = \mathrm{Com}$, $\Gamma(\mathrm{Com}, -) = \Gamma$ the free divided power monad !

Fact : If $\mathscr{P}(0) = 0$, then $\Gamma(\mathscr{P}, -)$ has a monad structure.

Divided power algebras over an operad

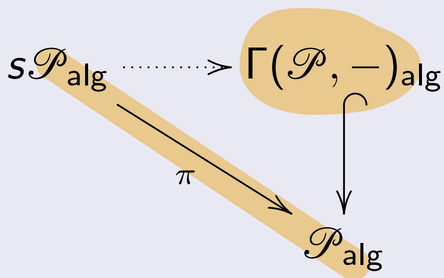
<u>Definition</u> : a divided power $\mathscr{P}$-algebra is a $\Gamma(\mathscr{P}, -)$-algebra.

Theorem (Fresse 2000)

For $\mathscr{P} = \mathrm{Lie}$ *: Divided power* Lie-*algebras are p-restricted Lie algebras.*

Theorem (Fresse 2000)

For any $\mathscr{P}$ *such that* $\mathscr{P}(0) = 0$,



Is there a general characterisation involving monomial operations ?

[F] Benoît Fresse, On the homotopy of simplicial algebras over an operad (2000).

Characterisation of divided power algebras

Fact 1 : A divided power algebra is a vector space A with a map

$$f_A : \Gamma(\mathscr{P}, A) \to A$$

satisfying relations (associativity and unity).

Fact 2 : $\Gamma(\mathscr{P}, A)$ is spanned by elements :

$$\mathfrak{t} := \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_s}} \sigma\mu \otimes \sigma\left(a_1^{\otimes r_1} \otimes \cdots \otimes a_s^{\otimes r_s}\right)$$

$= Sh(r_1, \ldots, r_s)$.

For $r_1 + \cdots + r_s = n$, $\mu \in \mathscr{P}(n)^{\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_s}}$, $a_i \in A$.

Construction : For all $\underline{r} = (r_1, \ldots, r_s)$, $\mu \in \mathscr{P}(r_1 + \cdots + r_s)^{\mathfrak{S}_{\underline{r}}}$, we obtain an *s*-ary monomial operation $\beta_{\mu,\underline{r}}$ such that :

$$\beta_{\mu,\underline{r}}(a_1, \ldots, a_s) = f_A(\mathfrak{t}).$$

These β's will play the role of the γ's of Cartan.

General characterisation of $\Gamma(\mathscr{P})$-algebras

Theorem (I.,2019)

A $\Gamma(\mathscr{P})$-algebra is a vector space A equipped with operations $\beta_{\mu,\underline{r}} : A^{\times s} \to A$, linear in μ, satisfying relations of the type : (S), (H), (M), (R/R), (L), (U), (C).

Upside 1 : This avoids complicated axiomatisation, and allows practical computation.

Upside 2 : This also ties the general case to the classical divided power algebra case and the restricted Lie algebra case.

Downside 1 : Ugly formulae.

Downside 2 : Too many operations.

Minimal characterisations

Toy example : Level algebras

Definition : A level algebra is a vector space A with a bilinear, commutative, non-necessarily associative operation $* : A \otimes A \to A$, such that :

$$(a * b) * (c * d) = (a * c) * (b * d).$$

Example : If A is a commutative associative algebra, $a * b := \lambda ab$ for all $\lambda \in \mathbb{F}$ makes $(A, *)$ into a level algebra.

Fact : There is an operad Lev whose algebras are level algebras.

Theorem (I. 2019)

Over a arbitrary field, the minimal characterisation of divided power Lev-algebras involves an infinite number of operations $\beta_{\mu,\underline{r}}$.

Theorem (I. 2021)

If char$(\mathbb{F}) = 2$, a divided power level algebra is a level algebra endowed with a "divided square".

$\mathscr{P}$-algebras with derivation

$d(ab) = d(a)b + ad(b)$.

Fact : For all $\mathscr{P}$ there is an operad $\mathrm{Der}_{\mathscr{P}}$ whose algebras are $\mathscr{P}$-algebra with a *$\mathscr{P}$-derivation* (following the examples As, Com, Lie,...)

Theorem (I. 2021)

A divided power $\mathrm{Der}_{\mathscr{P}}$*-algebra is a divided power $\mathscr{P}$-algebra* (A, β) *equipped with a linear map $d : A \to A$ satisfying :*

$$d\left(\beta_{\mu,\underline{r}}(a_1, \ldots, a_s)\right) = \sum_{i=1}^{s} \beta_{x,\underline{r}\circ_i(r_i-1,1)}(a_1, \ldots, a_i, d(a_i), a_{i+1} \ldots, a_s),$$

where $\underline{r} \circ_i (r_i - 1, 1) = (r_1, \ldots, r_{i-1}, r_i - 1, 1, r_{i+1}, \ldots, r_s)$.

For $\mathscr{P} = \mathrm{Com}$, we recover a notion of "*special derivation*" (Dokas, 2021), or "*power rule*" (Keigher–Pritchard, 1998) : $d(\gamma_n(x)) = d(x)\gamma_{n-1}(x)$
For $\mathscr{P} = \mathrm{Lie}$, we recover the notion of *restricted derivation* (Jacobson 1941).

Fact : If $\mathscr{P}$ is an operad, and A is a vector space with a linear $d : A \to A$, there is a universal way to extend d into a derivation on $S(\mathscr{P}, A)$.

Construction : Let $A = V \times V$ and $d(v_1, v_2) = (0, v_1)$. This induces a derivation d on $S(\mathscr{P}, V \times V)$. Restrict d to

$$d : S(\mathscr{P}, V) \to S(\mathscr{P}, V \times V),$$

For $\mathscr{P} = \mathrm{Com}$, we obtain the 'classical' differential combinator. We can do the same on $\Gamma(\mathscr{P}, V)$.

Question : To which condition on $\mathscr{P}$ is d a differential combinator ?

$$d : S(\mathscr{P}, V) \longrightarrow S(\mathscr{P}, V \times V)$$
$$\searrow \quad \Omega_{S(\mathscr{P},V)} \quad \nearrow$$

Connected to : [BLO] R. F. Blute, R. B. B. Lucyshyn-Wright, and K. O'Neill, Derivations in codifferential categories [2016]

Thank you for you attention.

(S) Symmetry :

$$\beta_{x,\underline{r}}((a_i)_i) = \beta_{\rho^*\cdot x,\underline{r}^\rho}((a_{\rho^{-1}(i)})_i) \quad \forall \rho \in \mathfrak{S}_p,$$

where ρ^* denotes the block permutation with blocks of size $(r_i)_{i\in[p]}$ associated to ρ.

(H) Homogeneity :

$$\beta_{x,(0,r_1,r_2,\ldots,r_p)}(a_0, a_1, \ldots, a_p) = \beta_{x,(r_1,r_2,\ldots,r_p)}(a_1, \ldots, a_p).$$

(M) Monomiality :

$$\beta_{x,\underline{r}}(\lambda a_1, a_2, \ldots, a_p) = \lambda^{r_1}\beta_{x,\underline{r}}(a_1, \ldots, a_p) \quad \forall \lambda \in \mathbb{F}.$$

(R/R) Reduction with respect to repetition of inputs :

$$\beta_{x,\underline{r}}(\underbrace{a_1,\dots,a_1}_{q_1},\underbrace{a_2,\dots,a_2}_{q_2},\dots,\underbrace{a_s,\dots,a_s}_{q_s})$$
$$= \beta_{\left(\sum_{\sigma\in\mathfrak{S}_{\underline{q}\triangleright\underline{r}}/\mathfrak{S}_{\underline{r}}}\sigma\cdot x\right),\ \underline{q}\triangleright\underline{r}}(a_1,a_2,\dots,a_s).$$

where $\underline{q}\triangleright\underline{r} = (r_1+\cdots+r_{q_1},\dots,r_{q_1+\cdots+q_{s-1}+1}+\cdots+r_p)$.

(L) Generalised Leibniz formula :

$$\beta_{x,\underline{r}}(a_0+a_1,\dots,a_p) = \sum_{l+m=r_1}\beta_{x,\underline{r}\circ_1(l,m)}(a_0,a_1,\dots,a_p),$$

where $\underline{r}\circ_1(l,m) = (l,m,r_2,r_3,\dots,r_p)$.

Ugly formulae

(U) Unitality :

$$\beta_{1_{\mathscr{P}},(1)}(a) = a \quad \forall a \in A.$$

(C) Composition :

$$\beta_{x,\underline{r}}(\beta_{x_1,\underline{q}_1}(\underline{b}_1),\ldots,\beta_{x_p,\underline{q}_p}(\underline{b}_p)) = \beta_{\sum_\tau \tau\cdot\mu\left(x\otimes\left(\bigotimes_{i=1}^{p} x_i^{\otimes r_i}\right)\right),\underline{r}\diamond(\underline{q}_i)_{i\in[p]}}(\underline{b}),$$

where I am obviously hiding some definitions under the rug.