Synthetic fibered $(\infty, 1)$-category theory

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Outline

Synthetic higher categories in type theory

- **Problem:** Weak infinite-dimensional categories: challenging to define and reason about analytically
- In particular: usually model-/presentation dependent and not *per se* homotopy invariant
- **A possible solution:** Use a simplicial extension of homotopy type theory due to Riehl–Shulman [RS17], also previously suggested by Joyal. This provides a synthetic account to (complete) Segal spaces (or Segal objects).
- Turns out, this captures well certain parts of $(\infty, 1)$-category theory.
- **In this talk:** Theory of co-/cartesian type families (Buchholtz-W), developed in sHoTT à la Riehl–Shulman [RS17] and Riehl–Verity's ∞-cosmos theory [RV22].
- Thus, obtain a synthetic theory of fibered $(\infty, 1)$-categories, even internal to any Grothendieck–Rezk–Lurie $(\infty, 1)$-topos due to Shulman's strictification of universes [Shu19].

Previous, supporting, and related work

- **On directed type theory and directed univalence:** Harper–Licata, Warren, Nuyts, Riehl–Shulman, Cavallo–Riehl–Sattler, Weaver–Licata, Buchholtz–W, Kudasov, Annenkov–Capriotti–Kraus–Sattler, Cisinski–Nguyen, North, Altenkirch–Sestini . . .
- **On fibrations of $(\infty, 1)$-categories:** Joyal, Lurie, Ayala–Francis, Barwick–Dotto–Glasman–Nardin–Shah, Rasekh, Riehl–Verity . . .
- **On Segal spaces and Segal objects/internal $(\infty, 1)$-categories:** Rezk, Joyal–Tierney, Lurie, Kazhdan–Varshavsky, Boavido de Brito, Rasekh, Martini . . .

Outline

Martin-Löf Type Theory (MLTT)

Martin-Löf type theory (Martin-Löf 1970s) is a system for **constructive logic.** It consists of **judgments** involving

$$\textbf{types } A, \quad \textbf{terms } a : A, \quad \textbf{contexts } \Gamma \equiv [x_1 : A_1, \ldots, x_n : A_n]$$

together with structural rules generating the derivations, as well as type formers Σ, Π, $\ldots$, and universe types $\mathcal{U}$.

There are **dependent types/type families** $\Gamma \vdash A$ (*e.g.* $n : \mathbb{N} \vdash \mathbb{R}^n$) and **dependent terms** $\Gamma \vdash f : A$ (*e.g.* $n : \mathbb{N} \vdash \vec{0}_n : \mathbb{R}^n$).

We have a notion of **judgmental equality** between types $A \equiv B$ and terms $a \equiv b$.

There also exists an internal, proof relevant notion of equality, captured by **propositional identity types**

$$x, y : A \vdash (x =_A y)$$

that support *path induction* (Martin–Löf's eliminator J). In particular, every A gives rise to:

$$\mathrm{refl}_{A,a} : (a =_A a), \quad \mathrm{inv}_{A,a,b} : (a =_A b) \to (b =_A a),$$
$$\mathrm{comp}_{A,a,b,c} : (b =_A c) \to (a =_A b) \to (a =_A c).$$

Homotopy Type Theory (HoTT)/Univalent Foundations (UF) I

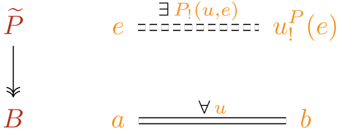
- Hofmann–Streicher 94: Groupoid model *refuting* Uniqueness of Identity Proofs (UIP) (LICS Test-of-Time Award 2014). Universe extensionality.
- Voevodsky, Streicher 06: Types as ∞-groupoids, *i.e.* Kan complexes.
- Voevodsky 06: internal definition of homotopy fibers, and weak equivalences. Univalent fibs (ext'ly) and h-levels. *Univalence Axiom:* $(A =_{\mathcal{U}} B) \simeq (A \simeq B) \rightsquigarrow$ *univalent foundations* for mathematics
- Awodey–Warren 06: Independently: MLTT as internal language of model categories, id types via path space fibrations. $\rightsquigarrow$ notion of *homotopy type theory* for synthetic homotopy theory (Awodey 07).
- van den Berg 06, van den Berg–Garner 08: Types as Batanin-ω-groupoids, internally
- Lumsdaine 09: Weak ω-categories from intensional type theory
- Garner 06, Gambino–Garner 08: Id types induce weak factorization system on syntactic categories
- Voevodsky, Kapulkin–Lumsdaine 12: Simplicial set model with univalent universes

Homotopy Type Theory (HoTT)/Univalent Foundations (UF) II

- Shulman 19: Every $(\infty, 1)$-topos gives rise to strict univalent universes, thereby confirming Awodey's Internal Language Conjecture that HoTT is an internal language of higher toposes.

Slogans:

- *Types are homotopy types/∞-groupoids.*
- *Dependent types are fibrations.*
- *Propositional identities are paths.*
- *Isomorphic types are equal.* (Voevodsky's Univalence Axiom)



$$\rightsquigarrow (-)_!^P : \prod_{a,b:B} (a =_B b) \to P(a) \to P(b)$$

sHoTT: Cubes, shapes, and topes

simplicial HoTT [RS17]: Multi-part contexts $\Xi \mid \Phi \mid \Gamma \vdash A$ with pre-type layers

1. **Abstract cubes (*cube layer*):** Lawvere theory generated by directed interval $\mathbb{2}$:

$$\frac{}{\mathbf{1}, \mathbb{2}\,\mathrm{cube}} \qquad \frac{}{\Xi \vdash \star : \mathbf{1}} \qquad \frac{}{\Xi \vdash 0, 1 : \mathbb{2}} \qquad \frac{I\,\mathrm{cube} \qquad J\,\mathrm{cube}}{I \times J\,\mathrm{cube}} \qquad \frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad [\ldots]$$

2. **Subpolytopes (*tope layer*):** Intuitionistic theory of formulas φ in cube contexts Ξ:

$$\frac{\varphi \in \Phi}{\Xi \mid \Phi \vdash \varphi} \qquad \frac{}{\Xi \vdash \bot, \top\,\mathrm{tope}} \qquad \frac{\Xi \vdash s : I \qquad \Xi \vdash t : I}{\Xi \vdash (s \equiv t)\,\mathrm{tope}} \qquad \frac{\Xi \vdash \varphi\,\mathrm{tope} \qquad \Xi \vdash \psi\,\mathrm{tope}}{\Xi \vdash (\varphi \wedge \psi), (\varphi \vee \psi)\,\mathrm{tope}}$$

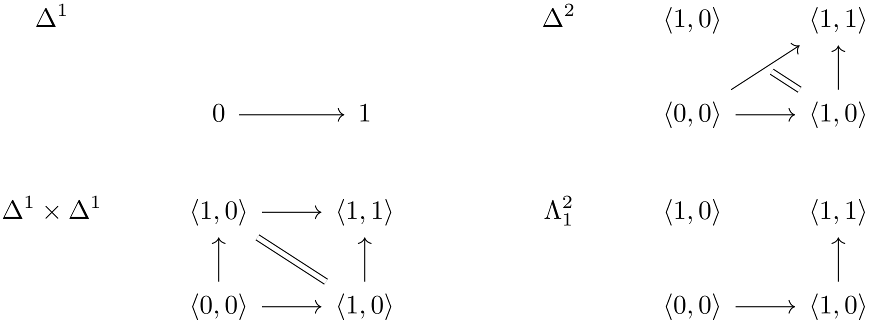
$$\frac{}{x, y : \mathbb{2} \vdash (x \leq y)\,\mathrm{tope}} \qquad [\ldots]$$

(including order axioms on $\mathbb{2}$)

Comprehension induces *shapes* (later, also promoted to *types*):

$$\frac{I\,\mathrm{cube} \qquad t : I \vdash \varphi\,\mathrm{tope}}{\{t : I \mid \varphi\}\,\mathrm{shape}}$$

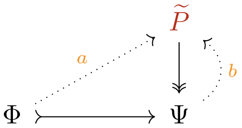
sHoTT: Examples of shapes



$$\Delta^1 :\equiv \{t : \mathbb{2} \mid \top\}, \quad \Delta^2 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\},$$
$$\Delta^1 \times \Delta^1 \equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid \top\}, \quad \Lambda^2_1 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid (s \equiv 0) \vee (t \equiv 1)\}$$

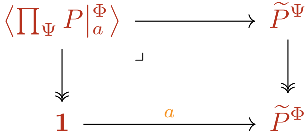
sHoTT: Extension types

Idea: Originally due to Lumsdaine–Shulman. For shape inclusions $\Phi \hookrightarrow \Psi$, families $P : \Psi \to \mathcal{U}$, and partial sections $a : \Pi_{t:\Phi} P(t)$ the ensuing *extension type* has inhabitants $b : \left\langle \prod_\Psi P \big|^{\Phi}_{a} \right\rangle$ capturing strict extensions of a ("Π-types with strict side conditions"):



$$b|_\Phi \equiv a$$

Semantically, modeled by pullback (slightly more involved over arbitrary type context Γ):



Strict extension types are weakly equivalent to their weak counterparts, but much easier to work with!

Outline

Hom types I

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

$$\hom_B(a,b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \Big|^{\partial \Delta^1}_{[a,b]} \right\rangle.$$

Definition (Dependent hom types, [RS17])

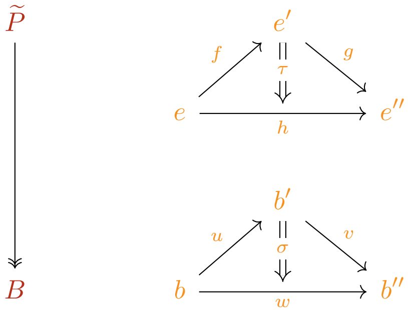
Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \Big|^{\partial \Delta^1}_{[d,e]} \right\rangle.$$

Hom types II

We will also be considering types of 2-cells: For arrows u, v, w in B with f, g, h in P lying above, with appropriate co-/domains, let

$$\hom^2_B(u,v;w) :\equiv \Big\langle \Delta^2 \to B \Big|^{\partial\Delta^2}_{[u,v,w]} \Big\rangle, \quad \mathrm{dhom}^{2,P}_{\sigma}(f,g;h) :\equiv \Big\langle \prod_{\langle t,s\rangle:\Delta^2} P(\sigma(t,s)) \Big|^{\partial\Delta^2}_{[f,g,h]} \Big\rangle.$$



Segal, Rezk, and discrete(=groupoidal) types I

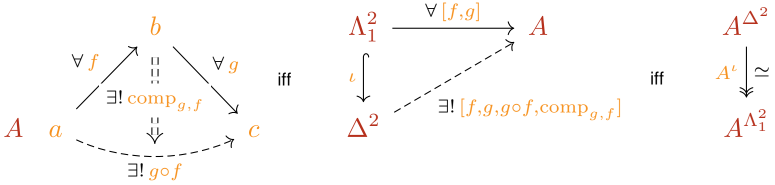
Definition (Synthetic ∞-categories, [RS17])

- *Synthetic pre-∞-category aka Segal type:* type that is local w.r.t. $\iota : \Lambda_1^2 \hookrightarrow \Delta^2$ (Joyal).
- *Synthetic ∞-category aka Rezk type:* Segal type A such that $\mathrm{idtoiso}_A : \prod_{x,y:A} (x =_A y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y)$, or equiv.: A is local w.r.t. either of the maps: $j_k : \mathbf{1} \hookrightarrow \mathbb{E}$ for $k = 0, 1$, or $!_{\mathbb{E}} : \mathbb{E} \to \mathbf{1}$ ($\mathbb{E}$ the free bi-inv. arrow)
- *Synthetic ∞-groupoid aka discrete type:* type A such that $\mathrm{idtoarr}_A : \prod_{x,y:A} (x =_A y) \xrightarrow{\simeq} \hom_A(x,y)$, or equiv.: A is local w.r.t. $i_0 : \mathbf{1} \hookrightarrow \Delta^1$.

Segal types have *weak composition of morphisms*:

$$\mathrm{isSegal}(B) \simeq \prod_{\kappa:\Lambda_1^2 \to B} \mathrm{isContr}\left(\left\langle \Delta^2 \to B \Big|_{\kappa}^{\Lambda_1^2} \right\rangle \right)$$

Segal, Rezk, and discrete(=groupoidal) types II



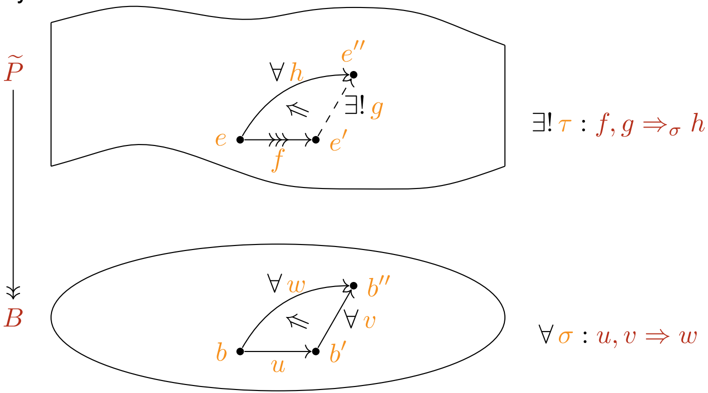
Arrow composition via the Segal condition

- Segal types have categorical structure: composition of morphisms, identites, and the corresponding laws (all up to homotopy)
- Functions between Segal types are automatically functorial.
- The hom-types of a Segal types are discrete.
- Rezk types are those Segal types that are, in addition, Rezk-complete/locally unvialent.
- Discrete types are those types all of whose arrows are invertible (automatically Rezk).
- Orthogonality characterizations imply closure properties.

Outline

Cocartesian arrows: Definition I

Intuitively: An arrow $f : e \to^P_u e'$ over $u : b \to_B b'$ is *cocartesian* if it satisfies the following universal property:



Cocartesian arrows: Definition II

Definition (Cocartesian arrows)

Let B be a type and $P : B \to \mathcal{U}$ be an inner family. Let $b, b' : B$, $u : \hom_B(b, b')$, and $e : P\, b$, $e' : P\, b'$. An arrow $f : \hom_{P\,u}(e, e')$ is a *(P-)cocartesian morphism* or *(P-)cocartesian arrow* iff

$$\mathrm{isCocartArr}_P\, f :\equiv \prod_{\sigma:\left\langle \Delta^2 \to B \middle|^{\Delta^1_0}_{u} \right\rangle} \; \prod_{h:\prod_{t:\Delta^1} P\,\sigma(t,t)} \mathrm{isContr}\left(\left\langle \prod_{\langle t,s\rangle:\Delta^2} P\sigma(t,s) \middle|^{\Lambda^2_0}_{[f,h]} \right\rangle\right).$$

Notice that being a cocartesian arrow is a proposition. Over a Segal base, this amounts to:

$$\mathrm{isCocartArr}_P\, f \simeq \prod_{b'':B} \; \prod_{v:\hom_B(b',b'')} \; \prod_{w:\hom_B(b,b'')} \; \prod_{\sigma:\hom^2_B(u,v;w)} \; \prod_{e'':P\,b''} \; \prod_{h:\mathrm{dhom}_{P\,w}(e,e'')} \mathrm{isContr}\left(\sum_{g:\mathrm{dhom}_{P\,v}(e',e'')} \mathrm{dhom}^2_{P\,\sigma}(f,g;h)\right)$$

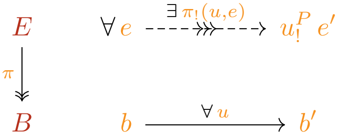
Cocartesian families: Definition

Definition (Cocartesian family)

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family such that $\widetilde{P}$ is a Rezk type. Then P is a *cocartesian family* if:

$$\mathrm{hasCocartLifts}\, P :\equiv \prod_{b,b':B} \prod_{u:b \to b'} \prod_{e:P\,b} \sum_{e':P\,b'} \sum_{f:e \to_u e'} \mathrm{isCocartArr}_P\, f$$

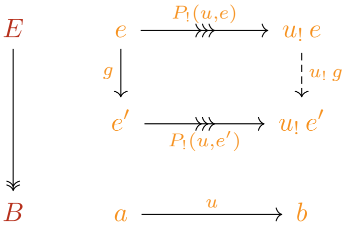
A map $\pi : E \twoheadrightarrow B$ is a *cocartesian fibration* iff $P :\equiv \mathrm{St}_B(\pi)$ is a cocartesian family.



$$\rightsquigarrow (-)_!^P : \prod_{a,b:B} (a \to_B b) \to P(a) \to P(b)$$

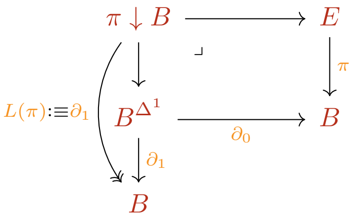
Cocartesian families: Functoriality

Hence, any $u : a \to_B b$ induces a functor $u_! : P\,a \to P\,b$ acting on arrows as follows:



Cocartesian families: Examples

1. For $g : C \to A \leftarrow B : f$, the comma projection $\partial_1 : f \downarrow g \twoheadrightarrow C$. (Hence, in particular the codomain projections $\partial_1 : A^{\Delta^1} \twoheadrightarrow A$.)
2. The *domain projection* $\partial_0 : A^{\Delta^1} \twoheadrightarrow A$, provided A has all pushouts.
3. For any map $\pi : E \to B$ between Rezk types, the *free cocartesian fibration*:



In particular, the desired UMP holds:

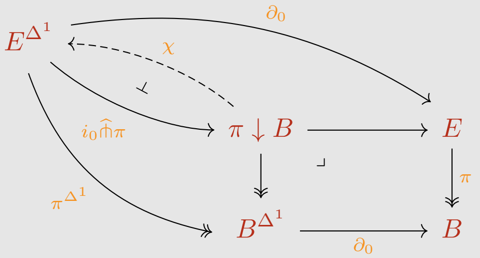
$$- \circ \iota : \mathrm{CocartFun}_B(L(\pi), \xi) \xrightarrow{\simeq} \mathrm{Fun}_B(\pi, \xi),$$

for any cocartesian fibration $\xi : F \to B$.

Cocartesian families: Characterization

Theorem (Chevalley criterion: Cocartesian families via lifting)

Let B be a Rezk type. A given isoinner family $P : B \to \mathcal{U}$ is cocartesian if and only if the Leibniz cotensor map $i_0 \pitchfork \pi : E^{\Delta^1} \to \pi \downarrow B$ has a left adjoint right inverse:



The idea is that $\chi : \pi \downarrow B \to E^{\Delta^1}$ is the *lifting map* $\chi(u, e) = P_!(u, e)$. Chevalley criterion implies a lot of closure properties!

Yoneda Lemma for cocartesian families I

Theorem (Dependent and absolute Yoneda Lemma)

1. ***Dependent Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $Q : b \downarrow B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} Q \xrightarrow{\simeq} Q(\mathrm{id}_b)$$

2. ***Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b as in*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} \partial_1^* P \xrightarrow{\simeq} P\, b$$

is an equivalence, where $\partial_1 : b \downarrow B \to B$.

Yoneda Lemma for cocartesian families II

This implies the discrete case from [RS17]. The proof uses the following proposition:

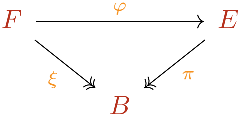
Proposition

Let B be a Rezk type, $b : B$ an initial object, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at b given by is an equivalence:

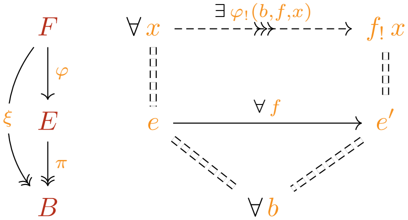
$$\mathrm{ev}_b : \prod_{B}^{\text{cocart}} P \xrightarrow{\simeq} P\,b$$

Sliced cocartesian families

For $\xi : F \twoheadrightarrow B$, $\pi : E \twoheadrightarrow B$, a fibered functor



is a *sliced cocartesian family* over B if:



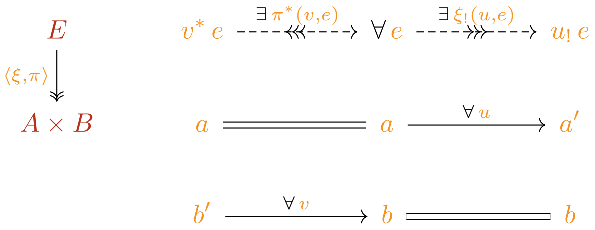
Externally, corresponds to cocartesian fibrations internal to Cat/B.

Two-sided cartesian families

A span

$$A \overset{\xi}{\twoheadleftarrow} E \overset{\pi}{\twoheadrightarrow} B \qquad \rightsquigarrow \qquad E \xrightarrow{\langle \xi, \pi \rangle} A \times B$$

is a *two-sided cartesian fibration* if



and the lifts *commute*, *i.e.* canonically

$$u_! \, v^* \, e =_{P(a,b)} v^* \, u_! \, e.$$

Developed in [W21].

Outline

Some perspectives

1. **Categorical universes and (multi-)modalities:** Rezk types Cat and Space. Modalities of cohesion ($\int \dashv \flat \dashv \sharp$) and direction ($\mathrm{op}$ and tw). Application: categorical Yoneda Lemma, involving the Yoneda embedding of type $A^{\mathrm{op}} \times A \to \mathrm{Space}$. Joint WIP w/ Buchholtz.
2. **Discrete two-sided fibrations:** More characterizations, closure properties, virtual equipment.
3. **Higher topos theory:** Use partial synthetic reasoning to extend (analytic) higher topos theory, *e.g.* Moens' Thm (cf. [W21]).
4. **Proof assistant:** Check out `rzk`, under development by Kudasov: `https://github.com/fizruk/rzk`

Outline

References

- U. Buchholtz, J. Weinberger (2021): Synthetic Fibered $(\infty, 1)$-Category Theory
arXiv:2105.01724
- N. Rasekh (2021): Cartesian Fibrations of Complete Segal Spaces
arXiv:2102.05190
- E. Riehl, M. Shulman (2017): A Type Theory for Synthetic ∞-Categories
Higher Structures **1**, no. 1, 147-–224.
- E. Riehl, D. Verity (2022): Elements of ∞-Category Theory
Book to appear at Cambridge University Press
- M. Shulman (2019): All $(\infty, 1)$-Toposes Have Strict Univalent Universes
arXiv:1904.07004
- T. Streicher (1999-2021): Fibered Categories à la Jean Bénabou
arXiv:1801.02927
- J. Weinberger (2021): A Synthetic Perspective on $(\infty, 1)$-Category Theory: Fibrational and Semantic Aspects. PhD thesis (submitted), Superv. Thomas Streicher, TU Darmstadt

Thank you!