Colimits of Higher Categories and Grothendieck Construction

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Introduction and dimension 1

From *SGA 4* [GSDV72], we have:

Pseudo-Colimits in Cat

Let $F\colon \mathcal{B} \to \mathbf{Cat}$ be a pseudo-functor from a category $\mathcal{B}$. Then

$$\varinjlim_{\mathcal{B}} F = \operatorname{el} F[\mathcal{C}^{-1}],$$

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And it's also a well-known fact that:

Lax Colimits in **Cat**

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$$\text{lax-}\varinjlim_{\mathcal{B}} F = \operatorname{el} F.$$

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Our goal here is to study similar results for *bicategories* to understand what happens when we have to deal with non-invertible higher cells.

We use Buckley's definition of fibration of bicategories and the corresponding Grothendieck construction [Buc14].

Exploring $\int$ in low dimensions

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$$f_{-}\colon Ff(x_{-}) \to y_{-}$$

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$$f_-\colon Ff(x_-) \to y_-$$

- **2-cells:** Pairs $(\theta, \theta_-)\colon (f, f_-) \to (g, g_-)$ with $\theta\colon f \to g$ and

$$\begin{tikzcd}
Ff(x_-) \arrow[r, "f_-"] \arrow[d, "F\theta(x_-)"'] \arrow[dr, phantom, "\Downarrow\theta_-"] & y_- \\
Fg(x_-) \arrow[ur, "g_-"'] &
\end{tikzcd}$$

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Lax Grothendieck Construction

For $F, G \colon \mathcal{B} \to \mathbf{Bicat}$ trihomomorphisms from a bicategory, we have a pseudo-functor:

$$\int : [\mathcal{B}, \mathbf{Bicat}]_{\mathrm{lax}}(F, G) \longrightarrow [\textstyle\int F, \int G]_{\mathrm{lax}/\mathcal{B}}$$

(where $[\mathcal{B}, \mathbf{Bicat}]_{\mathrm{lax}}$ is a tricategory of trihomomorphisms, lax trinatural transformations, lax modifications and perturbations, and where $[\int F, \int G]_{\mathrm{lax}/\mathcal{B}}$ is the bicategory of pseudo-functors, lax-natural transformations and modifications lying strictly over $\mathcal{B}$)

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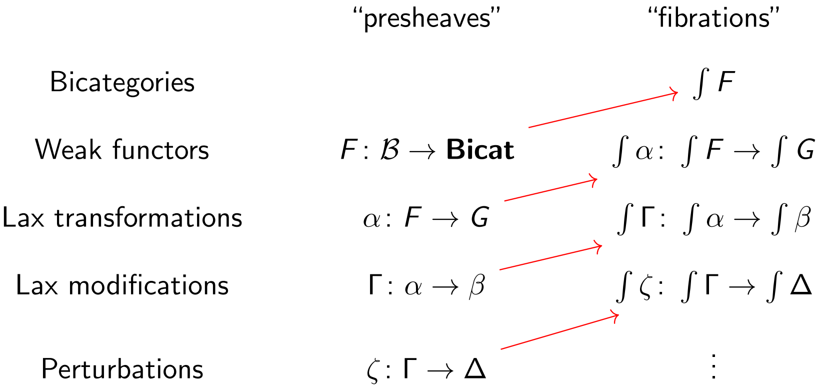

	“presheaves”	“fibrations”
Bicategories		$\int F$
Weak functors	$F\colon \mathcal{B} \to \mathbf{Bicat}$	$\int \alpha\colon \int F \to \int G$
Lax transformations	$\alpha\colon F \to G$	$\int \Gamma\colon \int \alpha \to \int \beta$
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We could introduce *lax* diagrams $F\colon \mathcal{B} \to \mathbf{Bicat}$. However *weak* on LHS corresponds to *fibrations* on RHS (and the following result would fail if G is too lax).

Local equivalence of the *Lax* construction

Almost-theorem: local *lax* equivalence

The functor $\int : [\mathcal{B}, \mathbf{Bicat}]_{\mathsf{lax}}(F, G) \longrightarrow [\int F, \int G]_{\mathsf{lax}/\mathcal{B}}$ described above is a biequivalence.

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…asking for "weak" transformations on LHS corresponds to asking for "Cartesian" functors on the RHS…

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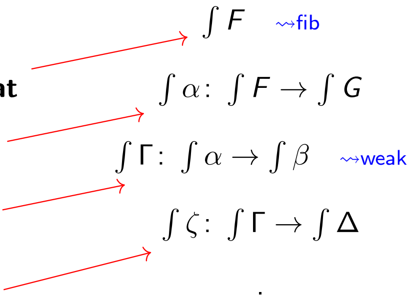

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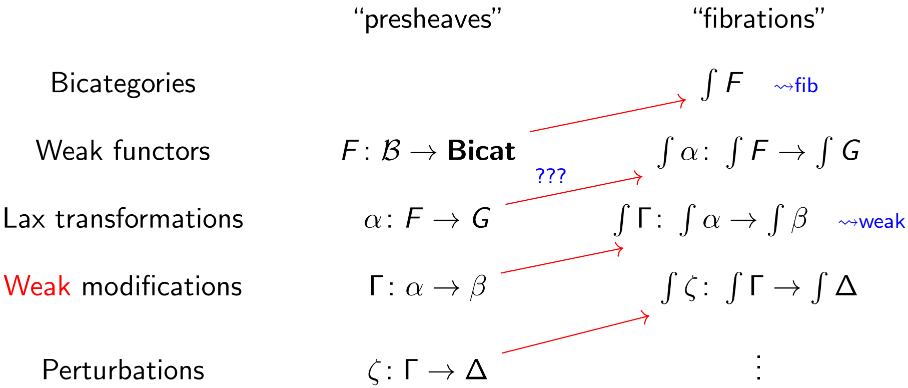

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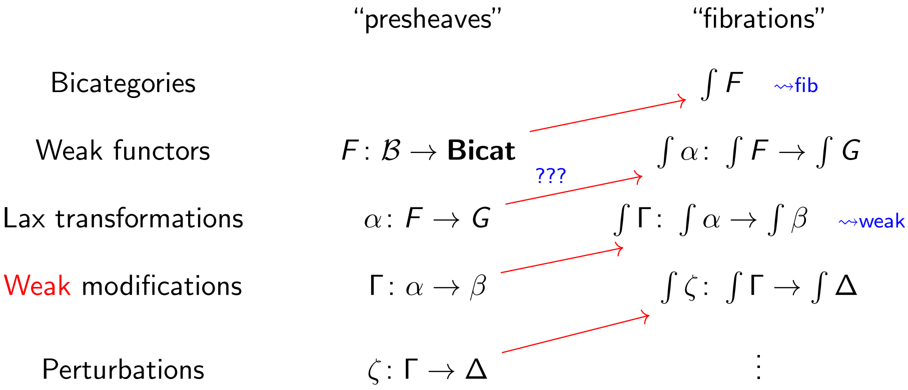

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From the third level up, *lax*/*weak* on LHS corresponds to *lax*/*weak* on RHS.

Correspondence at the level of transformations

Under the *restricted* biequivalence of the previous slide, we have, for $\alpha\colon F \to G$,

$$\begin{aligned} \alpha_f \text{ is an equivalence} \quad &\iff \quad \textstyle\int \alpha \text{ respects Cartesian arrows above } f \\ \alpha_\theta \text{ is an equivalence} \quad &\iff \quad \textstyle\int \alpha \text{ respects Cartesian 2-cells above } \theta \end{aligned}$$

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Theorem: local σ equivalence

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- This is easier to define than weighted tricolimits and *should* be equivalent (as illustrated in the last slide).
- In between *pseudo* and *lax*: can choose exactly which coherence cells of the cocones are equivalences and which are oriented.

Computing higher colimits of higher categories

We then have a very short proof of how to compute colimits:

Key observation

The Grothendieck construction of a constant diagram $\Delta(\mathcal{X})$ is the product $\mathcal{X} \times \mathcal{B}$.

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$$\begin{aligned}[\mathcal{B}, \mathbf{Bicat}]_\sigma(F, \Delta(-)) &\simeq \sigma\text{-Fib}(\textstyle\int F, \int \Delta(-)) \\ &\simeq \sigma\text{-Fib}(\textstyle\int F, \mathcal{B} \times -) \\ &\simeq [\textstyle\int F, -]_{\mathcal{C}_\Sigma} \\ &\simeq [\textstyle\int F[\mathcal{C}_\Sigma^{-1}], -]\end{aligned}$$

where $\mathcal{C}_\Sigma$ is the collection of Cartesian 1- and 2-cells above Σ and $[\int F, \mathcal{X}]_{\mathcal{C}_\Sigma}$ is the full sub-bicategory of pseudo-functors sending cells of $\mathcal{C}_\Sigma$ to equivalences.

Generalizations from *SGA 4*

In particular, for $F\colon \mathcal{B} \to$ **Bicat** a trihomomorphism, we can generalize the two results we started from:

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These are the two extreme cases of σ-tricolimits. Arbitrary σ-tricolimits should be as expressive as weighted tricolimits.

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We expect a similar proof for tricategories with a universal property of a tricategorical comma.

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- In the first two papers, we show that the localization at only 1-cells is already enough to compute all filtered conical tricolimits in **Bicat**, observing a curious phenomenon: the 2-cells are localized "for free".

References

John D. Berman.
On Lax Limits in ∞-Categories, 2020.

Mitchell Buckley.
Fibred 2-Categories and Bicategories.
Journal of Pure and Applied Algebra, 218(6):1034–1074, June 2014.

P. Carrasco, A. M. Cegarra, and A. R. Garzón.
Classifying Spaces for Braided Monoidal Categories and Lax Diagrams of Bicategories.
Advances in Mathematics, 226(1):419–483, January 2011.

Maria Emilia Descotte, Eduardo J. Dubuc, and Martin Szyld.
A Construction of Certain Weak Colimits and an Exactness Property of the 2-Category of Categories.
Theory and Applications of Categories, 33(8):192–215, March 2018.

David Gepner, Rune Haugseng, and Thomas Nickelsen Nikolaus.
Lax Colimits and Free Fibrations in ∞-Categories.
Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, 22:1255–1266, 2017.

Robert Gordon, A. John Power, and Ross Street.
Coherence for Tricategories.
Number no. 558 in American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society, 1995.

John W. Gray.
Formal Category Theory: Adjointness for 2-Categories.
Lecture Notes in Mathematics. Springer Berlin Heidelberg, Berlin New York, 1 edition, 1974.

Alexander Grothendieck, Bernard Saint-Donat, and Jean-Louis Verdier.
Théorie des Topos et Cohomologie Étale des Schémas, Tome 2, volume 270 of *Lecture Notes in Mathematics*.
Springer-Verlag Berlin Heidelberg, Berlin New York, 1972.

Dorette A. Pronk.
Étendues and Stacks as Bicategories of Fractions.
Compositio Mathematica, 102(3):243–303, 1996.