Lecture

Back to Lambda-Calculus

Remember we worked with λ-terms like:

$$\lambda x.xy$$
$$\lambda x.xx$$
$$\lambda x\lambda y.yx$$
etc

The main operation is application.

Intuition: We are applying a function to a term

$$[\lambda x.f(x)]M = f(M) \leftarrow \beta\text{-reduction}$$

Result, often called untyped λ-calculus.

"untyped": means anything can be applied to anything.

So you wind up with oddities like:

$$(\lambda x.xx)(\lambda x.xx)$$ which reduces to itself

Its reduction graph looks like:

In particular, the reduction process does not terminate. (There are more complicated examples with more complicated graphs. (See Barendregt.)

This is either great or awful, depending on your viewpoint.

If we want to use λ-calculus as the denotations of proofs, it's terrible.

Why? Application corresponds to the cut rule in logic

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \text{cut}$$

Also called composition by L&S

The cut-elimin process always terminates.

Solution: Add Types

See p 72 of L&S

The idea: Terms will have types. If M is of type A, write $M:A$.

The only terms that can be applied to $M:A$ are terms of the form $A \Rightarrow B$. The result of the application will be a term of type B.

Terms like $\lambda x.xx$ cannot be given a ~~const~~ consistent type.

In fact, none of the problems of untyped λ-calculus arise.

Types: 1) There are 2 basic types
$\mathbb{1}$ and $\mathbb{N}$ (natural #s)
2) If A and B are types,
so are $A \times B$ and B^A (I'll write $A \Rightarrow B$)
There can be other types in more complicated settings. There can be other operations, like list(A).

Terms: 1) Every type has a countably infinite set of variables. If x is of type A, we'll write x^A or $x:A$ or $x \in A$
2) The type $\mathbb{1}$ has a special term $* \in \mathbb{1}$.
3) If $a \in A$, $b \in B$, $\langle a, b \rangle \in A \times B$
4) If $c \in A \times B$, then
$\pi(c) \in A$ $\pi'(c) \in B$

5) If $f \in B^A$ and $a \in A$, then

~~f~~ $f`a \in B$.

Note the weird notation.

6) If $x \in A$ is a variable and $\varphi \in B$, then $\lambda x.\varphi \in B^A$

7) $0 \in N$, if $n \in N$, then $S(n) \in N$

8) $a \in A$, $h \in A^A$, $n \in N$, then
$I(a,h,n) \in A$.

This is the iteration operator. Think of this as

$$h^n(a)$$

Note it only makes sense for $h \in A^A$

Next one develops a theory of equality for λ-terms. There are annoying technical details.

But in the end, we have the following results:

1) Every typed λ-calculus generates a cartesian closed category with $\mathbb{N}$

2) Every cartesian closed category with $\mathbb{N}$ determines a typed λ-calculus, its <u>internal language</u>.

These 2 constructions are functorial and the 2 categories are equivalent to one another.

Details on p. 72-80 of L&S.

3) There are decision procedures for determining when 2 λ-terms are equal

Back to untyped λ-calculus

Something like xx can't be modeled in the typed λ-calculus.

x would have to be a function type, say B^A. But then x would have to also be of type A.

But suppose we could construct a cartesian closed category with an object A such that $A \cong A^A$.

We would also need $A \cong A \times A$.

The terminal object 1 always satisfies these equations. But this isn't interesting.

Thm: In Set, the only such object is 1

Proof: By cardinality.

Such an object is called reflexive.

First construction due to Dana Scott.

It's since been simplified. This is in Section 18 of L&S.

Def'n: An ω-poset is a poset in which every countable ascending chain of elements has a supremum.

Morphisms: order-preserving maps that preserve sups of ascending chains

Explicitly, $f: M \to N$ such that

1) $x \leq y$ in $M \Rightarrow f(x) \leq f(y)$ in N

2) If $a_0 \leq a_1 \leq a_2 \ldots$ is an ascending chain, then it has a sup, denoted $\bigvee_i a_i$

We have $f(\bigvee_i a_i) = \bigvee_i f(a_i)$

This category has lots of applications

In computer science. <u>Domain Theory</u>

<u>EX:</u> Let P be an ω-poset with a least element, $\perp$. Consider a map $f: P \to P$

$$p = \perp \leq f(\perp) \leq f^2(\perp) \leq \ldots$$

Then p is a fixed point, i.e. $f(p) = p$.

In fact it is the least fixed point of f

We'll call the category ω-Pos

Thm: ω-Pos is a cartesian closed category.

Proof: Products are constructed pointwise, with pointwise ordering.

The internal hom $P \Rightarrow Q$ is

$$\mathrm{Hom}_{\omega\text{-Pos}}(P, Q)$$

with pointwise ordering. Then define

$$f \leq g \text{ iff } \forall p \in P, \ f(p) \leq g(p)$$

It just remains to check the adjunction.
You can just write down the unit & counit.

Thm: There is an object $V \in \omega\text{-Pos}$
V has a least element $\perp$ and $V \neq 1$
and $V \cong V \times V$

Proof: Let A be any nontrivial object with a least element. Let $V = \prod_{i=1}^{\infty} A$
be a product of countably ∞ many copies of A. It's useful to think of V as $V = A^{\mathbb{N}}$.

Define $\alpha, \beta : V \to V$ by
$$\alpha(v)(n) = v(2n) \qquad \beta(v)(n) = v(2n+1)$$

Then $\sigma = \langle \alpha, \beta \rangle : V \to V \times V$ is an iso
$\sigma^{-1} : V \times V \to V$ is defined by

$$\sigma^{-1}(v_1, v_2)(2n) = v_1(n)$$

$$\sigma^{-1}(v_1, v_2)(2n+1) = v_2(n)$$

Easy to check. $\square$

$V \cong V^V$ is much more complicated.

One of the issues is that V^V is not functorial in V. Given

$f: V \to W$, I can't construct a map $V^V \to W^W$ or $W^W \to V^V$.

Make sure you understand this.

Def Let $A, B \in \omega\text{-POS}$. A <u>decreasing retraction</u> $A \to B$ is a pair

$f: A \to B$ and $g: B \to A$

such that $g \circ f = id_A$, $f \circ g \leq id_B$

These form a category $\omega\text{-POS}^{\#}$

The contravariant endofunctor

$$\mathrm{Hom}_{\omega\text{-Pos}}(-,V) = V^{(\ )} : \omega\text{-Pos} \to \omega\text{-Pos}$$

induces a covariant endofunctor

$F_V : \omega\text{-Pos}^\# \to \omega\text{-Pos}^\#$

If I have $(f,g) : A \to B$ in $\omega\text{-Pos}^\#$

define $F_V(f,g) = (V^g, V^f) : V^A \to V^B$

Make sure you get this!

$$\frac{g : B \to A}{V^g : V^A \to V^B} \qquad \frac{f : A \to B}{V^f : V^B \to V^A}$$

Details need checking

General categorical lemma: Lemma 18.6

Let $\mathcal{C}$ be any category such that every diagram of the following form has a colimit

$$A_0 \to A_1 \to A_2 \to \cdots$$

Let $F: \mathcal{C} \to \mathcal{C}$ be a functor that preserves such colimits. Let $h: V \to F(V)$ be any arrow. Consider

$$V \xrightarrow{h} F(V) \xrightarrow{F(h)} F^2(V) \to \text{etc} \qquad *$$

If U is the colimit of this sequence, then

$$U \cong F(U)$$

Why? Applying F to $*$, I get

$$F(V) \to F^2 V \to \text{etc}$$

which is the tail of $*$.

Thm 13.7 of L&S

$\exists$ an object $U \in \omega$-POS s.t. $U \not\cong 1$

and $U \times U \cong U \cong U^U$

Proof: Let A be any non-trivial ω-pos

Let $V = A^{\mathbb{N}}$

Construct an arrow $(f,g) : V \to F_V^{\#}(V) = V^V$

defined by
$$f : V \to V^V$$
$$g : V^V \to V$$

$$f(v)(w) = v \qquad g(h) = h(0)$$

Need to check $g \circ f = 1_V$ $\quad f \circ g \leq 1_{V^V}$ ✓

Next need to check ω-POS$^{\#}$ satisfies the conditions of 18.6.

Then check F_V preserves them.

Let U be the colimit of the sequence of lemma 18.6 So we have

$$U \cong F_V(U) = V^U$$

Now calculate

$$U \times U \cong V^U \times V^U \cong (V \times V)^U \cong V^U \cong U$$

and

$$U^U \cong (V^U)^U \cong V^{U \times U} \cong V^U \cong U. \quad \square$$

Equations like

$$U \cong U^U$$

are examples of <u>recursive domain equations</u>

Finding the appropriate categorical settings is important area of research.