# A brief journey from Bicategories to Cartesian linear bicategories

Shayesteh Naeimabadi

Graduate student of
Richard Blute and
Pieter Hofstra

University of Ottawa

November 13, 2021

# Content

**Bicategories**
Jean Bénabou - 1967

**Cartesian Bicategories**
Carboni and Walters - 1987
Carboni, Kelly, Walters, and Wood - 2008

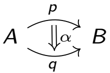
**Linear Bicategories**
Cockett, Koslowski, and Seely - 2000

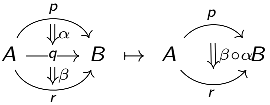
**Cartesian Linear Bicategories**

# Overview

- Bicategories are a weak version of 2-categories, where the horizontal composition is associative up to coherent isomorphism, and similarly for identities.
- The definition of cartesian bicategory axiomiatizes important examples of bicategories such as **Rel**, the category of sets and relations.
- Linear bicategories are a generalization of bicategories which introduce a second composition, and the two compositions are linked by linear distributions as suggested by linear logic.
- Our goal is to extend the definition of cartesian bicategory to the linear setting.
- **Rel** is not just a bicategory, it is a linear bicategory. So it should be our first example of a cartesian linear bicategory.

A *bicategory* $\mathcal{B}$ consists of

- A collection of objects (0-cells): $A$, $B$, $C$, ...
- For each pair of objects $(A, B)$, a (hom-) category $\mathcal{B}(A, B)$ with
  - Objects (1-cells): $A \xrightarrow{p} B$,
  - Arrows (2-cells): 

- Vertical composition $\circ$: 

- For each object $A$, an identity functor

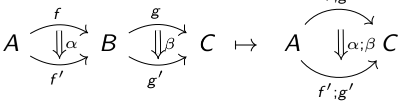
$$I_A : 1 \to \mathcal{B}(A, A)$$

# Bicategories

- A horizontal composition (bi)functor ;

$$;_{A,B,C} : \mathcal{B}(A,B) \times \mathcal{B}(B,C) \to \mathcal{B}(A,C)$$

$$A \xrightarrow{f} B \xrightarrow{g} C \mapsto \; f;g : A \longrightarrow C$$




where it is associative up to coherent isomorphism, and similarly has two unitors.

# Example **Rel**

- 0-cells are sets: $A, B, C, \ldots$
- Each (hom-) category $\mathcal{B}(A,B)$ is $\mathcal{P}(A \times B)$ which is a poset under inclusion with
  - 1-cells are relations $R : A \to B$
  - 2-cells $A \underset{R'}{\overset{R}{\Downarrow \alpha}} B$ are $\alpha : R \subseteq R'$
- Composition (bi)functor is an ordinary composition between relations.

$$;_{A,B,C} : \mathcal{B}(A,B) \times \mathcal{B}(B,C) \to \mathcal{B}(A,C)$$

$$A \xrightarrow{R} B \xrightarrow{S} C \mapsto R;S := \{(a,c) \mid \exists b \in B \ \ (a,b) \in R \text{ and } (b,c) \in S\}$$

$$A \underset{S}{\overset{R}{\Downarrow \alpha}} B \underset{S'}{\overset{R'}{\Downarrow \beta}} C \ \mapsto \ A \underset{R';S'}{\overset{R;S}{\Downarrow \alpha;\beta}} C \ \ (R;S \subseteq R';S')$$

- Identity $I_A : 1 \to \mathcal{B}(A,A)$

# Example ($Q$-**Rel** where $Q$ is a quantale)

First we recall definition of a quantale:

## Definition (Quantale)

A quantale is a complete lattice $Q$ which carries a monoid structure with neutral element $e$ with an associative binary operation $* : Q \times Q \to Q$, satisfying a distributive property

$$a * \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a * b_i)\,, \qquad \bigvee_{i \in I} a_i * b = \bigvee_{i \in I} (a_i * b)$$

For example: The quantale generated by a monoid $\mathcal{M} = (M, e, .)$ is $\mathcal{PM} = (\mathcal{P}M, \subseteq, \{e\}, .)$, where $\mathcal{P}M$ is the power set of $M$.

# Example ($Q$-**Rel** where $Q$ is a quantale)

- 0-cells are sets $A, B, C, \ldots$
- Since we can order $Q$-relations pointwise by using the quantale ordering

$$R \subseteq R' \iff \forall a, b \ \ R(a,b) \leq R'(a,b)$$

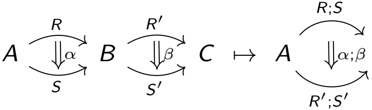
each (hom-) category $\mathcal{B}(A,B)$ is a poset under inclusion with

  - 1-cells are $Q$-relations $R : A \to B$ ($R : A \times B \to Q$)
  - 2-cells are $\alpha : R \subseteq R'$

- Composition (bi)functor is

$$;_{A,B,C} : \mathcal{B}(A,B) \times \mathcal{B}(B,C) \to \mathcal{B}(A,C)$$

$$(R,S) \ \mapsto \ R;S(a,c) := \bigvee_{b \in B} R(a,b) * S(b,c)$$

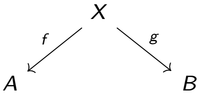



$(R; S \subseteq R'; S')$

- Identity $I_A : 1 \to \mathcal{B}(A,A)$ where $I : A \to A$ is

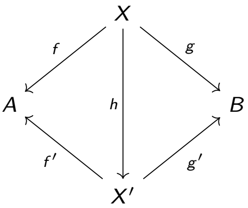
$$I(a,a') = \begin{cases} e & \text{if } a = a' \\ \bot & \text{if } a \neq a' \end{cases}$$

- 0-cells are objects of $\mathcal{C}$
- Categories Span$(A, B)$ with
  - 1-cells are $(f, g) : A \to B$




$(X, f, g)$

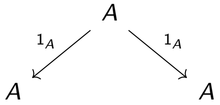
  - 2-cells are $h : X \to X'$

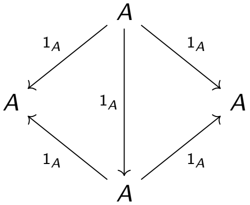



such that $f' \circ h = f$ and $g' \circ h = g$. Composition in Span$(A, B)$ is inherited from $\mathcal{C}$.

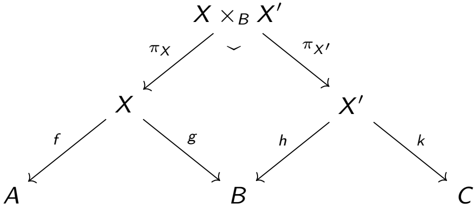
# Example Span($\mathcal{C}$)

- Identity $I_A : 1 \to \text{Span}(A, A)$




- Composition (bi)functor is defined by using pullback as follows




$$(X, f, g); (X', h, k) = (X \times_B X', f \circ \pi_X, k \circ \pi_{X'}).$$

Horizontal composition of 2-cells is defined by the universal property of pullbacks.

# Homomorphism between bicategories

Let $\mathcal{B}$ and $\mathcal{B}'$ are two bicategories. A homomorphism $T : \mathcal{B} \to \mathcal{B}'$ consists of

- A function $T$ mapping objects of $\mathcal{B}$ into objects of $\mathcal{B}'$
- For each pair $(A, B)$ of objects, a functor $T_{A,B} : \mathcal{B}(A, B) \to \mathcal{B}'(TA, TB)$
- For each triple $(A, B, C)$ of objects, a natural transformation

$$\begin{array}{ccc} \mathcal{B}(A,B) \times \mathcal{B}(B,C) & \xrightarrow{;_{A,B,C}} & \mathcal{B}(A,C) \\ {\scriptstyle T_{A,B} \times T_{B,C}} \downarrow & \overset{m_{A,B,C}}{\nearrow\!\!\!\nearrow} & \downarrow {\scriptstyle T_{A,C}} \\ \mathcal{B}(TA,TB) \times \mathcal{B}'(TB,TC) & \xrightarrow[;_{TA,TB,TC}]{'} & \mathcal{B}'(TA,TC) \end{array}$$

- For each object $A$, a natural transformation

$$u : I'_{TA} \Longrightarrow T_{A,A} \circ I_A$$

satisfying coherence axioms.

**Remark**

In this section, bicategory $\mathcal{B}$ is a locally posetal bicategory.

## Remark

In this section, bicategory $\mathcal{B}$ is a locally posetal bicategory.

## Definition

A tensor product in $\mathcal{B}$ is a homomorphism of bicategories

$$\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$$

together with an object $I$, called the identity object, and natural isomorphisms

$$\rho : X \to X \otimes I$$
$$\gamma : X \otimes Y \to Y \otimes X$$
$$\alpha : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

satisfying a number of coherence conditions. That is, we get a symmetric monoidal structure on $\mathcal{B}$.

# Cartesian bicategory

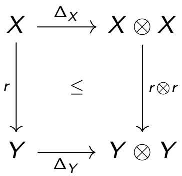
A *cartesian* structure on a bicategory $\mathcal{B}$ consists of

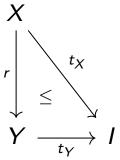
- A tensor product in $\mathcal{B}$
- On every object $X$ in $\mathcal{B}$, a cocommutative comonoid structure. That is, arrows

$$\Delta_X : X \to X \otimes X, \qquad t_X : X \to I.$$

These data must satisfy the following axioms
(U) Each arrow $r : X \to Y$ is a lax comonoid homomorphism. That is,

$$r; \Delta_Y \leq \Delta_X; (r \otimes r) \qquad \text{and} \qquad r; t_Y \leq t_X$$




(M) Comultiplication $\Delta_X$ and counit $t_X$ have right adjoins $\Delta_X^*$, $t_X^*$.

# Cartesian bicategory

## Definition

An arrow $r : X \to Y$ in a bicategory $\mathcal{B}$ is called a *map* if it has a right adjoint $r^*$. Denote $\mathbf{Map}(\mathcal{B})$ the subbicategory of $\mathcal{B}$ determined by these maps.

## Definition

An arrow $r : X \to Y$ in a bicategory $\mathcal{B}$ is called a *map* if it has a right adjoint $r^*$. Denote $\mathbf{Map}(\mathcal{B})$ the subbicategory of $\mathcal{B}$ determined by these maps.

## Theorem (Carboni and Walters)

Let $\mathcal{B}$ be a locally posetal bicategory. If $\mathcal{B}$ has a Cartesian structure, then

1. $\mathbf{Map}(\mathcal{B})$ has finite products,
2. Each hom-category $\mathcal{B}(X, Y)$ has finite products which is denoted by $\wedge$.
3. For any 1-cells $r$ and $s$ we have the following formula in $\mathcal{B}$:

$$r \otimes s = (p^*; r; p) \wedge (q^*; s; q) \qquad (p \text{ and } q \text{ are appropriate projections})$$

Conversely, if $\mathbf{Map}(\mathcal{B})$ staisfies (i) and (ii) and the formula in (iii) defines a functorial tensor product on $\mathcal{B}$, then $\mathcal{B}$ has a cartesian structure.

# Linear bicategories (Cockett, Koslowski, Seely)

A *linear bicategory* $\mathcal{B}$ consists of

- A collection of objects (0-cells): $A,\ B,\ C,\ \ldots$
- For each pair of objects $(A, B)$, a (hom-) category $\mathcal{B}(A, B)$
- Two composition (bi)functors

$$;_{A,B,C}\ ,\ \bullet_{A,B,C}\ : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$$

  and units $\top_A, \bot_A \in \mathcal{B}(A, A)$, each coherently associate and unital. That is, we have two bicategory structures $(;, \top)$ and $(\bullet, \bot)$.

- Natural transformations $\delta_L$ and $\delta_R$, called *linear distributivities*

$$\begin{array}{ccc} \mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D) & \xrightarrow{Id \times \bullet} & \mathcal{B}(A, B) \times \mathcal{B}(B, D) \\ {\scriptstyle ; \times Id}\downarrow & \overset{\delta_L}{\Longleftarrow} & \downarrow{\scriptstyle ;} \\ \mathcal{B}(A, C) \times \mathcal{B}(C, D) & \xrightarrow[\bullet]{} & \mathcal{B}(A, D) \end{array}$$

  and,

$$
\begin{array}{ccc}
\mathcal{B}(A,B)\times\mathcal{B}(B,C)\times\mathcal{B}(C,D) & \xrightarrow{\bullet\times Id} & \mathcal{B}(A,B)\times\mathcal{B}(B,D) \\
{\scriptstyle Id\times ;}\downarrow & \overset{\delta_R}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle ;} \\
\mathcal{B}(A,C)\times\mathcal{B}(C,D) & \xrightarrow[\bullet]{} & \mathcal{B}(A,D)
\end{array}
$$

These must satisfy several coherence conditions.

# Example (**Rel**)

- 0-cells are sets, $A, B, C, \ldots$
- (hom-) category $\mathcal{B}(A, B)$ is $\mathcal{P}(A \times B)$ which is a poset under inclusion with
  - 1-cells are relations $R : A \nrightarrow B$
  - 2-cells $A \underset{R'}{\overset{R}{\Downarrow \alpha}} B$ are $\alpha : R \subseteq R'$
- Two Composition (bi)functors

$$;_{A,B,C} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$$

$$A \xrightarrow{R} B \xrightarrow{S} C \mapsto R; S := \{(a, c) |\ \exists b \in B \ \ (a, b) \in R \ \wedge \ (b, c) \in S\}$$

$$A \underset{R'}{\overset{R}{\Downarrow \alpha}} B \underset{S'}{\overset{S}{\Downarrow \beta}} C \ \mapsto \ A \underset{R';S'}{\overset{R;S}{\Downarrow \alpha;\beta}} C \quad (R; S \subseteq R'; S')$$

$$\bullet_{A,B,C} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$$

$$A \xrightarrow{R} B \xrightarrow{S} C \mapsto R \bullet S := \{(a, c) |\ \forall b \in B \ \ (a, b) \in R \text{ or } (b, c) \in S\}$$

$$R \bullet S \subseteq R' \bullet S'$$

# Linear functors

Let $\mathcal{B}$ and $\mathcal{B}'$ are two linear bicategories. A linear functor $F : \mathcal{B} \to \mathcal{B}'$ consists of

- A function $F$ mapping objects of $\mathcal{B}$ into objects of $\mathcal{B}'$
- For each pair $(A, B)$ of objects, two functors $F_{;}, F_{\bullet} : \mathcal{B}(A, B) \to \mathcal{B}'(FA, FB)$
- 2-cells $m_{\top} : \top_{F_{;}(A)} \Longrightarrow F_{;(\top_A)}$ and $n_{\perp} : F_{\bullet_{\perp_A}(A)} \Longrightarrow \perp_{F_{\bullet}(A)}$
- Natural transformations, which with $m_{\top}$ and $n_{\perp}$ make $F_{;}$ monoidal (or "lax") with respect to **;** and $F_{\bullet}$ comonoidal (or "colax") with respect to $\bullet$

$$m_{;} : F_{;}(A);F_{;}(B) \Longrightarrow F_{;}(A;B)$$

  and similarly for $\bullet$.
- Natural transformations (called "linear strengths")

$$\nu_{;}^{R} : F_{;}(A \bullet B) \Longrightarrow F_{\bullet}(A) \bullet F_{;}(B)$$
$$\nu_{;}^{L} : F_{;}(A \bullet B) \Longrightarrow F_{;}(A) \bullet F_{\bullet}(B)$$

  and similarly for $\bullet$.

satisfying some coherence axioms.

# Linear adjunction

The following can be seen as a lemma or definition of linear adjunction.

**Lemma (Cockett, Koslowski, Seely)**

Given 1-cells $A : X \to Y$, $B : Y \to X$ in a linear bicategory, the following are equivalent

1. $A$ is left linear adjoint to $B$
2. For all 0-cells $Z$, the functor $(-);A : Hom(Z, X) \to Hom(Z, Y)$ is left adjoint to the functor $(-) \bullet B$
3. For all 0-cells $Z$, the functor $B;(-) : Hom(X, Z) \to Hom(Y, Z)$ is left adjoint to the functor $A \bullet (-)$

# Cartesian linear bicategories

## Remark

In this section, linear bicategory $\mathcal{B}$ is a locally posetal linear bicategory.

## Remark

In this section, linear bicategory $\mathcal{B}$ is a locally posetal linear bicategory.

## Definition

A tensor product and a cotensor product in $\mathcal{B}$ are linear functors of linear bicategories

$$\otimes, \oplus : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$$

together with two symmetric monoidal structures, denoted $(\mathcal{B}, \otimes, ;, \top)$ and $(\mathcal{B}, \oplus, \bullet, \bot)$, respectively. And, for each triple $(X, Y, Z)$ of objects, the category $\mathcal{B}$ must also be equipped with two distributive natural transformations:

$$\delta_L : X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus Z$$
$$\delta_R : (X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)$$

satisfying a number of coherence conditions.

# Cartesian linear bicategory

A *cartesian* structure on a locally posetal linear bicategory $\mathcal{B}$ consists of

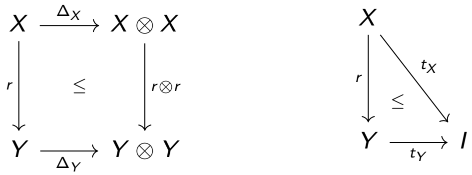
- A tensor $\otimes$ product and a cotensor product $\oplus$ in $\mathcal{B}$.
- On every object $X$ in $\mathcal{B}$, a cocommutative comonoid structure, and a commutative monoid structure. That is, arrows

$$\Delta_X : X \to X \otimes X, \qquad t_X : X \to \top,$$

$$\nabla_X : X \oplus X \to X, \qquad \epsilon_X : \bot \to X.$$

satisfy the following axioms
($U_1$) Each arrow $r : X \to Y$ is a colax comonoid homomorphism. That is

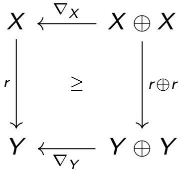
$$r;\Delta_Y \leq \Delta_X;(r \otimes r) \qquad \text{and} \qquad r;t_Y \leq t_X$$

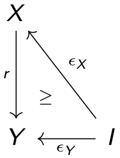


# Cartesian linear bicategory

($U_2$) Each arrow $r : X \to Y$ is a lax monoid homomorphism. That is

$$(r \oplus r) \bullet \nabla_Y \leq \nabla_X \bullet r \qquad \text{and} \qquad \epsilon_X \bullet r \leq \epsilon_Y$$




(M) Comultiplication $\Delta_X$ and counit $t_X$ have right linear adjoints.

# Example (**Rel**)

- Compositions $R;S := \{(x,z) | \ \exists y \in Y \ \ (x,y) \in R \ \wedge \ (y,z) \in S\}$ and $R \bullet S := \{(x,z) | \ \forall y \in Y \ \ (x,y) \in R \ \vee \ (y,z) \in S\}$
- $(\mathbf{Rel}, \otimes, \mathbf{;}, \top)$ is a symmetric monoidal category
  - Define $\otimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ on objects $X \otimes Y := X \times Y$
  - Define $\otimes$ on morphisms by $R \otimes S : X \otimes X' \to Y \otimes Y'$ where $(x,x')R \otimes S(y,y')$ iff $(x,y) \in R \wedge (x',y') \in S$
- $(\mathbf{Rel}, \oplus, \bullet, \bot)$ is a symmetric monoidal category
  - Define $\oplus$ on objects $X \oplus Y := X \times Y$
  - Define $\oplus$ on morphisms by $R \oplus S : X \oplus X' \to Y \oplus Y'$ where $(x,x')R \oplus S(y,y')$ iff $(x,y) \in R \vee (x',y') \in S$

# Example

- Two distributive natural transformations:

$$\delta_L : X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus Z$$
$$\delta_R : (X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)$$

- For every object $X$ we have a cocommutative comonoid and a commutative monoid structure. That is, arrows $\Delta_X : X \to X \otimes X$ and $t_X : X \to \top$ and their duals $\nabla_X : X \oplus X \to X$ and $\epsilon_X : \bot \to X$

- Inequalities

$$r;\Delta_Y \leq \Delta_X;(r \otimes r) \qquad \text{and} \qquad r;t_Y \leq t_X$$
$$(r \oplus r) \bullet \nabla_Y \leq \nabla_X \bullet r \qquad \text{and} \qquad \epsilon_X \bullet r \leq \epsilon_Y$$

- Comultiplication $\Delta_X$ and counit $t_X$ have right linear adjoints by using the recent Lemma.

# Future works

- Find and study more examples of cartesian linear bicategories
  - Bicategory of relations **Rel**($\mathcal{E}$) in a regular category $\mathcal{E}$?
  - Ordered objects and ideals?

- Will have a theorem similar to Carboni and Walters's describing cartesian structure on **Map(**$\mathcal{B}$**)**?

- Can we define cartesian linear bicategories in general not just locally posetal?

# Bibliography


*E. Aleiferi. Cartesian Double Categories with an Emphasis on Characterizing Spans*, https://arxiv.org/abs/1809.06940. 2018.

*J.R.B Cockett, J. Koslowski and R.A.G Seely . Introduction to Linear bicategories*. Math. Struct. Comp. Science, 10(2): 165-203 (2000).

*A. Carboni and R.F.C Walters. . Cartesian Bicategories I*. Journal of Pure and Applied Algebra, 49:11–32 (1987).

*T. Fox. Coalgebras and cartesian categories*. Comm. Algebra 4(7):665-667 (1976).

*Jean Benabou. Introduction to bicategories*. Springer, Berlin, 1967.

Thank you!