Schur functors and categorified plethysm

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Schur functors and categorified plethysm, joint with John Baez and Todd Trimble, arXiv:2106.00190, 2021.

- the ring Λ
- birings, plethories
- 2-rigs
- abstract Schur functors
- 2-birigs, 2-plethories
- main theorem

The ring Λ

Fundamental Theorem of Representations of Symmetric Groups

Isomorphism classes of irreducible representations of S_n are in bijection with partitions of n.

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$V, W \in \mathrm{Rep}(\mathrm{S})$

$$(V \otimes W)_n := \sum_{i+j=n} V_i \otimes W_j$$

Denote the ring $K_0(\mathrm{Rep}(\mathrm{S}))$ by Λ.

The ring Λ

Of the representation ring of the symmetric groups Λ, Hazewinkel [Haz09] writes:

> *It seems unlikely that there is any object in mathematics richer and/or more beautiful than this one [....]*

It has a very rich structure. We seek to shed light on the origin of this structure.

Birings

(ring = commutative and unital)

Definition ([TW70])

A **biring** *is a ring object in* $(\mathsf{Ring}^{\mathrm{op}}, \otimes, \mathbb{Z})$.
$\mathsf{Biring} = \mathsf{Ring}(\mathsf{Ring}^{\mathrm{op}})^{\mathrm{op}}$.

ring homomorphisms

- coaddition: $\alpha\colon \mathsf{B} \to \mathsf{B} \otimes \mathsf{B}$
- co-zero: $o\colon \mathsf{B} \to \mathbb{Z}$
- conegation: $\nu\colon \mathsf{B} \to \mathsf{B}$
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co-associative, co-unital, co-distributive

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Example

- $\mathbb{Z}$ is a biring with $\alpha = \mu\colon \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \otimes \mathbb{Z}$,
  $o = \epsilon = \nu = 1_{\mathbb{Z}}$.
- $\mathbb{Z}[x]$ is more interesting:
  $\alpha(x) = x \otimes 1 + 1 \otimes x$
  $o(x) = 0$
  $\nu(x) = -x$
  $\mu(x) = x \otimes x$
  $\epsilon(x) = 1$
- $\Lambda \ldots$

Biring lifts

B $\in$ Biring, R $\in$ Ring, then the set $\mathrm{Ring}(B, R)$ gets a ring structure: for $f, g \in \mathrm{Ring}(B, R)$, $f + g$ is the composite

$$B \xrightarrow{\alpha} B \otimes B \xrightarrow{f \otimes g} R \otimes R \xrightarrow{\nabla} R$$

and $f * g$ is the composite

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$$\begin{tikzcd} & \mathrm{Ring} \arrow[d, "U"] \\ \mathrm{Ring} \arrow[ur, dashed, "\Phi_B"] \arrow[r, "\mathrm{Ring}(B,-)"'] \arrow[r, phantom, shift left=3, "\Rightarrow\sim"] & \mathrm{Set} \end{tikzcd}$$

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- Ring$(\mathbb{Z}, R) = *$, so $\Phi_{\mathbb{Z}} = 0$.
- Ring$(\mathbb{Z}[x], R) = UR$, so $\Phi_{\mathbb{Z}[x]} = id_{\text{Ring}}$.
- Λ represents the big Witt vector functor.

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Proposition

The following categories are equivalent.

- Biring
- $\mathrm{LAdj}(\mathrm{Ring}, \mathrm{Ring})$
- $\mathrm{RAdj}(\mathrm{Ring}, \mathrm{Ring})^{\mathrm{op}}$

Plethories

$$
\begin{tikzcd}
\text{Ring} \arrow[r, bend right, "\Phi_B"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right=60, "\Phi_{B'}\circ\Phi_B=\Phi_{B'\odot B}"'] \arrow[rr, phantom, bend right=30, "\perp"] & \text{Ring} \arrow[l, bend right] \arrow[r, bend right, "\Phi_{B'}"'] \arrow[r, phantom, "\perp"] & \text{Ring} \arrow[l, bend right] \arrow[ll, bend right=60]
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Definition ([TW70, BW05])

A **plethory** *is a monoid in* (Biring, $\odot$)*. Equivalently, a plethory is a biring P such that the lift Φ_P is a right adjoint comonad.*

biring homomorphisms
$p\colon P \odot P \to P$ and $u\colon \mathbb{Z}[x] \to P$
associative, unital

Example

- $\mathbb{Z}$ is not a plethory!
- $\Phi_{\mathbb{Z}[x]} = 1_{\mathrm{Ring}}$ is a comonad, so $\mathbb{Z}[x]$ is a plethory, $p =$ polynomial comp.
- Λ is a plethory, as we shall see.

2-rigs

Definition

A **2-rig** *is a symmetric monoidal Cauchy complete linear category*

- $\otimes \colon R \times R \to R$, $I \colon 1 \to R \ldots$
- $\mathrm{Vect}_{\mathbb{C}}$-enriched
- has absolute colimits (biproducts $\oplus$, a zero object, and all idempotents split)

Natural $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ automatically.

Example

- FinVect is the initial 2-rig.
- reps of a group on vector spaces
- G-graded vector spaces for a group
- bounded chain complexes of vector spaces
- vector bundles over a topological space
- algebraic vector bundles over an algebraic variety
- coherent sheaves over an algebraic variety or scheme or alg stack

Free 2-rig on 1

What is the free 2-rig on 1?

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, bend left, "\mathsf{S}"] \arrow[r, phantom, "\perp"] & \mathsf{SMC} \arrow[l, bend left, "U_0"] \arrow[r, bend left, "k(-)"] \arrow[r, phantom, "\perp"] & \mathsf{SMLin} \arrow[l, bend left, "U_1"] \arrow[r, bend left, "\overline{(-)}"] \arrow[r, phantom, "\perp"] & \text{2-Rig} \arrow[l, bend left, "U_2"] \end{tikzcd}$$

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Theorem

$\mathrm{Rep}(\mathsf{S}) \simeq \overline{k\mathsf{S}}$.

- functors $F \colon \mathsf{S} \to \mathsf{Vect}$ extend to linear functors $\tilde{F} \colon k\mathsf{S} \to \mathsf{Vect}$

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- functors $F \colon \mathsf{S} \to \mathsf{Vect}$ extend to linear functors $\tilde{F} \colon k\mathsf{S} \to \mathsf{Vect}$
- representables are the regular representations
- all irreps are retracts of regular reps

2-birigs, 2-plethories

Definition

A **2-birig** *is a 2-rig* B *with a lift of* $\mathsf{2\text{-}Rig}(B, -)$ *through* $U \colon \mathsf{2\text{-}Rig} \to \mathsf{Cat}$.

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\end{tikzcd}$$

Definition

A **2-plethory** *is a 2-birig* P *such that* Φ_{P} *is a 2-(right adjoint comonad).*

Example

- FinVect is the initial 2-rig, so it represents the constant 0 2-functor. This is not a 2-comonad though, so FinVect is not a 2-plethory.
- As the free 2-rig on 1, Rep(S) represents the identity 2-functor. The identity 2-functor on is a 2-comonad, so Rep(S) is a 2-plethory.

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- As the free 2-rig on 1, $\mathsf{Rep}(\mathsf{S})$ represents the identity 2-functor. The identity 2-functor on is a 2-comonad, so $\mathsf{Rep}(\mathsf{S})$ is a 2-plethory.

Main Theorem (Baez–Moeller–Trimble)

The 2-plethory structure on $\mathsf{Rep}(\mathsf{S})$ *induces the plethory structure on* $K(\mathsf{Rep}(\mathsf{S})) \cong \Lambda$.

John C. Baez, Joe Moeller, and Todd Trimble.
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