

# Schur functors and categorified plethysm

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*Schur functors and categorified plethysm*,  
joint with John Baez and Todd Trimble,  
arXiv:2106.00190, 2021.

- ▶ the ring  $\Lambda$
- ▶ birings, plethories
- ▶ 2-rigs
- ▶ abstract Schur functors
- ▶ 2-birigs, 2-plethories
- ▶ main theorem



# The ring $\Lambda$

## Fundamental Theorem of Representations of Symmetric Groups

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We could instead consider

$$S \simeq \text{FinBij} \simeq \prod_n BS_n$$

$$\text{Rep}(S) \cong \bigoplus_n \text{Rep}(S_n)$$

$$V, W \in \text{Rep}(S)$$

$$(V \otimes W)_n := \sum_{i+j=n} V_i \otimes W_j$$

Denote the ring  $K_0(\text{Rep}(S))$  by  $\Lambda$ .

# The ring $\Lambda$

Of the representation ring of the symmetric groups  $\Lambda$ , Hazewinkel [Haz09] writes:

*It seems unlikely that there is any object in mathematics richer and/or more beautiful than this one [...]*

It has a very rich structure. We seek to shed light on the origin of this structure.



# Birings

(ring = commutative and unital)

## Definition ([TW70])

A **biring** is a ring object in  $(\text{Ring}^{\text{op}}, \otimes, \mathbb{Z})$ .

$\text{Biring} = \text{Ring}(\text{Ring}^{\text{op}})^{\text{op}}$ .

ring homomorphisms

- ▶ coaddition:  $\alpha: B \rightarrow B \otimes B$
- ▶ co-zero:  $o: B \rightarrow \mathbb{Z}$
- ▶ conegation:  $\nu: B \rightarrow B$
- ▶ comultiplication:  $\mu: B \rightarrow B \otimes B$
- ▶ co-one:  $\epsilon: B \rightarrow \mathbb{Z}$

co-associative, co-unital, co-distributive

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## Example

- ▶  $\mathbb{Z}$  is a biring with  $\alpha = \mu: \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \otimes \mathbb{Z}$ ,  
 $o = \epsilon = \nu = 1_{\mathbb{Z}}$ .
- ▶  $\mathbb{Z}[x]$  is more interesting:  
 $\alpha(x) = x \otimes 1 + 1 \otimes x$   
 $o(x) = 0$   
 $\nu(x) = -x$   
 $\mu(x) = x \otimes x$   
 $\epsilon(x) = 1$
- ▶  $\Lambda \dots$

## Biring lifts

$B \in \text{Biring}$ ,  $R \in \text{Ring}$ , then the set  $\text{Ring}(B, R)$  gets a ring structure: for  $f, g \in \text{Ring}(B, R)$ ,  $f + g$  is the composite

$$B \xrightarrow{\alpha} B \otimes B \xrightarrow{f \otimes g} R \otimes R \xrightarrow{\nabla} R$$

and  $f * g$  is the composite

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A commutative diagram illustrating the relationship between the categories  $\mathbf{Ring}$  and  $\mathbf{Set}$  and the lifting of multiplication. The diagram consists of three nodes:  $\mathbf{Ring}$  at the top left,  $\mathbf{Ring}$  at the bottom left, and  $\mathbf{Set}$  at the bottom right. A solid arrow labeled  $\mathbf{Ring}(B, -)$  points from the bottom-left  $\mathbf{Ring}$  to the bottom-right  $\mathbf{Set}$ . A dashed arrow labeled  $\Phi_B$  points from the bottom-left  $\mathbf{Ring}$  to the top-left  $\mathbf{Ring}$ . A solid arrow labeled  $\sim$  points from the top-left  $\mathbf{Ring}$  to the bottom-right  $\mathbf{Set}$ . A solid arrow labeled  $U$  points from the top-left  $\mathbf{Ring}$  to the bottom-right  $\mathbf{Set}$ .

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### Example

- ▶  $\mathbf{Ring}(\mathbb{Z}, R) = *$ , so  $\Phi_{\mathbb{Z}} = 0$ .
- ▶  $\mathbf{Ring}(\mathbb{Z}[x], R) = UR$ , so  $\Phi_{\mathbb{Z}[x]} = id_{\mathbf{Ring}}$ .
- ▶  $\Lambda$  represents the big Witt vector functor.

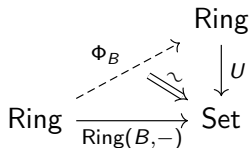
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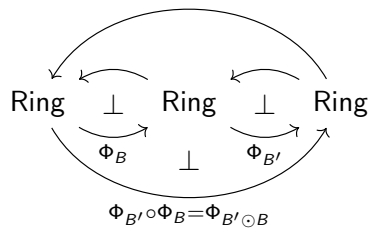
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### Proposition

*The following categories are equivalent.*

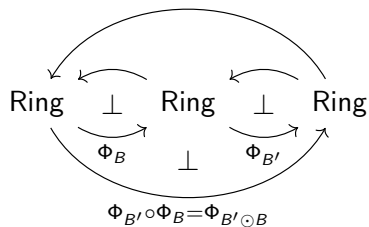
- ▶  $\mathbf{Biring}$
- ▶  $\mathbf{LAdj}(\mathbf{Ring}, \mathbf{Ring})$
- ▶  $\mathbf{RAdj}(\mathbf{Ring}, \mathbf{Ring})^{\text{op}}$

# Plethories



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## Definition ([TW70, BW05])

A **plethory** is a monoid in  $(\mathbf{Biring}, \odot)$ .  
Equivalently, a plethory is a biring  $P$  such that the lift  $\Phi_P$  is a right adjoint comonad.

biring homomorphisms

$$p: P \odot P \rightarrow P \text{ and } u: \mathbb{Z}[x] \rightarrow P$$

associative, unital

## Example

- ▶  $\mathbb{Z}$  is not a plethory!
- ▶  $\Phi_{\mathbb{Z}[x]} = 1_{\mathbf{Ring}}$  is a comonad, so  $\mathbb{Z}[x]$  is a plethory,  $p = \text{polynomial comp.}$
- ▶  $\Lambda$  is a plethory, as we shall see.



# 2-rigs

## Definition

A **2-rig** is a symmetric monoidal Cauchy complete linear category

- ▶  $\otimes: R \times R \rightarrow R$ ,  $I: 1 \rightarrow R \dots$
- ▶  $\text{Vect}_{\mathbb{C}}$ -enriched
- ▶ has absolute colimits (biproducts  $\oplus$ , a zero object, and all idempotents split)

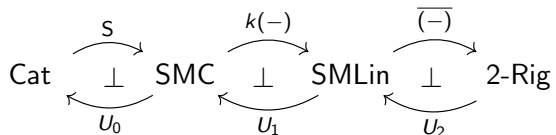
Natural  $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$  automatically.

## Example

- ▶  $\text{FinVect}$  is the initial 2-rig.
- ▶ reps of a group on vector spaces
- ▶  $G$ -graded vector spaces for a group
- ▶ bounded chain complexes of vector spaces
- ▶ vector bundles over a topological space
- ▶ algebraic vector bundles over an algebraic variety
- ▶ coherent sheaves over an algebraic variety or scheme or alg stack

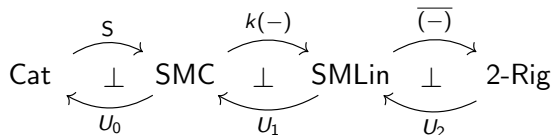
## Free 2-rig on 1

What is the free 2-rig on 1?



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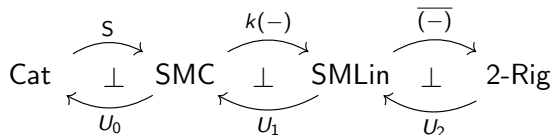
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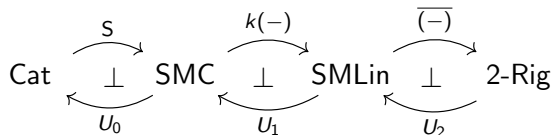
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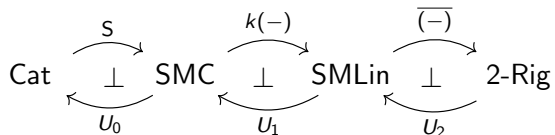
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- ▶  $S(1) \simeq S$
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- ▶  $\overline{kS} \subseteq \text{Vect}^{kS^{\text{op}}}$  on retracts of direct sums of representables.

# Free 2-rig on 1

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## Theorem

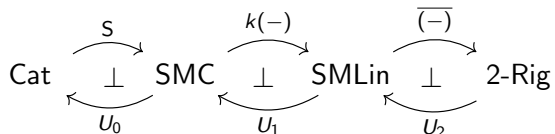
$$\text{Rep}(S) \simeq \overline{kS}.$$

- ▶ functors  $F: S \rightarrow \text{Vect}$  extend to linear functors  $\tilde{F}: kS \rightarrow \text{Vect}$

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- ▶ functors  $F: S \rightarrow \text{Vect}$  extend to linear functors  $\tilde{F}: kS \rightarrow \text{Vect}$
- ▶ representables are the regular representations
- ▶ all irreps are retracts of regular reps

## 2-birigs, 2-plethories

### Definition

A **2-birig** is a 2-rig  $B$  with a lift of  $2\text{-Rig}(B, -)$  through  $U: 2\text{-Rig} \rightarrow \text{Cat}$ .

A commutative diagram illustrating the relationship between the 2-Rig and Cat categories. The diagram consists of four nodes: a '2-Rig' node at the top left, a '2-Rig' node at the top right, a '2-Rig' node at the bottom left, and a 'Cat' node at the bottom right. An arrow labeled  $\Phi_B$  points from the bottom-left '2-Rig' node to the top-right '2-Rig' node. An arrow labeled  $2\text{-Rig}(B, -)$  points from the bottom-left '2-Rig' node to the bottom-right 'Cat' node. A vertical arrow labeled  $U$  points from the top-right '2-Rig' node to the bottom-right 'Cat' node. A diagonal arrow labeled  $\sim$  (with two parallel lines) points from the top-right '2-Rig' node to the bottom-right 'Cat' node, indicating a natural isomorphism.



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The diagram shows a commutative triangle. The top vertex is labeled  $2\text{-Rig}$ . The bottom-left vertex is also labeled  $2\text{-Rig}$ . The bottom-right vertex is labeled  $\text{Cat}$ . An arrow labeled  $\Phi_B$  points from the bottom-left  $2\text{-Rig}$  to the top  $2\text{-Rig}$ . An arrow labeled  $U$  points from the top  $2\text{-Rig}$  to  $\text{Cat}$ . An arrow labeled  $2\text{-Rig}(B, -)$  points from the bottom-left  $2\text{-Rig}$  to  $\text{Cat}$ . A double arrow with a tilde symbol  $\sim$  connects the arrow  $\Phi_B$  and the arrow  $U$ , indicating a natural isomorphism between the two paths from  $2\text{-Rig}$  to  $\text{Cat}$ .

### Definition

A **2-plethory** is a 2-birig  $P$  such that  $\Phi_P$  is a 2-(right adjoint comonad).

### Example

- ▶  $\text{FinVect}$  is the initial 2-rig, so it represents the constant 0 2-functor. This is not a 2-comonad though, so  $\text{FinVect}$  is not a 2-plethory.
- ▶ As the free 2-rig on 1,  $\text{Rep}(S)$  represents the identity 2-functor. The identity 2-functor on is a 2-comonad, so  $\text{Rep}(S)$  is a 2-plethory.

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### Main Theorem (Baez–Moeller–Trimble)

The 2-plethory structure on  $\text{Rep}(S)$  induces the plethory structure on  $K(\text{Rep}(S)) \cong \Lambda$ .



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