

# Determinant functors and the K-theory of tensor triangulated categories

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November 13, 2021

Background:

nice category (Waldhausen)  $\mathcal{W}$



K-theory spectrum



compute  $k_0$  and  $k_1$   
associated Picard groupoid  $\mathcal{P}$

$$k_0(\mathcal{W}) = \pi_0 \mathcal{P}$$

$$k_1(\mathcal{W}) = \pi_1 \mathcal{P}$$

symmetric monoidal cat.  
+ all morphisms are invertible

one wants to get  $\mathcal{P}$ :  
universal determinant  
functor

$$\det: \mathcal{W} \rightarrow \mathcal{P}$$

will work for  $\mathcal{T}$

Triangulated Category  $\mathcal{T}$

- additive category

- equivalence

- class of diagrams called distinguished triangle

$$\Delta: x \rightarrow y \rightarrow z \rightarrow \Sigma x$$

$$\Sigma: \mathcal{T} \rightarrow \mathcal{T} \text{ (shift)}$$

$$\begin{array}{ccc} x & \rightarrow & y \\ +1 \uparrow & & \downarrow z \end{array}$$

+ axioms

ex  
derived category  
abelian cat

# Universal determinant functor on $\mathcal{T}$

det functor

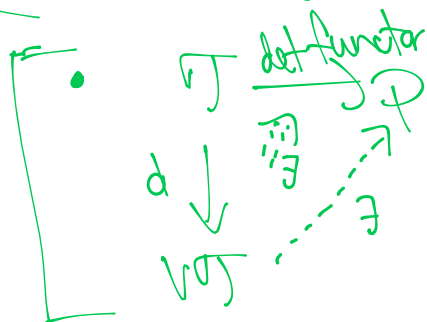
$$d: \mathcal{T} \rightarrow \mathcal{V}\mathcal{T}$$

triangulated cat

picard groupoid

- a functor  $\text{isom}(\mathcal{T}) \rightarrow \mathcal{V}\mathcal{T}$
- for any distinguished triangle  $x \rightarrow y \rightarrow z \rightarrow \Sigma x$   
 $d(z) = d(x) + d(y)$   
subject to some compatibility

universal



## Tensor Triangulated Category

- triangulated category
- tensor category  $\otimes: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$   
unit, associate, +
- compatibility between the 2 structure  
e.g.

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \xrightarrow{\otimes} & \mathcal{T} \\ \uparrow \cong & & \uparrow \cong \\ \Sigma(x \otimes y) & & (\Sigma x) \otimes y \end{array}$$

→ can be viewed as a bunch of

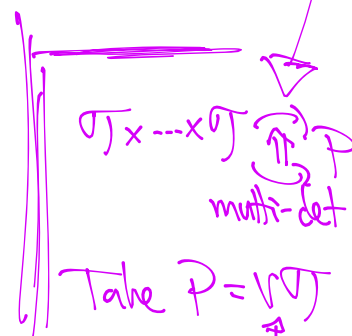
$$\mathcal{T} \times \dots \times \mathcal{T} \xrightarrow{\otimes} \mathcal{T} \xrightarrow{d} \mathcal{P}$$

i.e. info is encoded in the multi-cat. of triangulated cat. (2-cat)

## Our extension: Multi-determinant functor

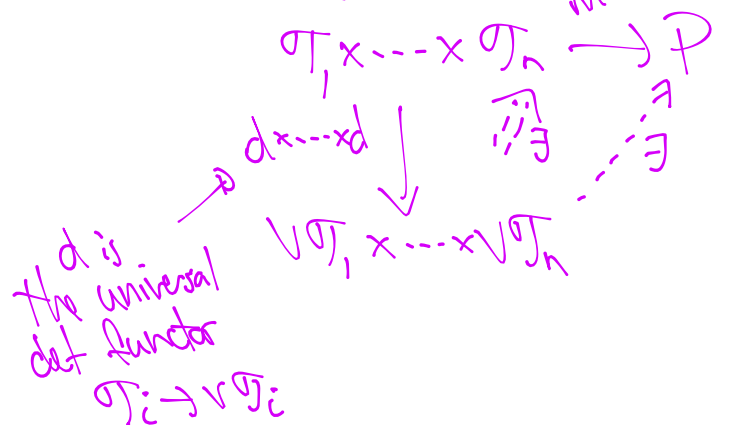
$$d: \mathcal{T}_1 \times \dots \times \mathcal{T}_n \rightarrow \mathcal{P}$$

- functor  $d: \text{isom}(\mathcal{T}_1) \times \dots \times \text{isom}(\mathcal{T}_n) \rightarrow \mathcal{P}$
- determinant in each variable with pairwise compatibility



Theorem (and A.)

From un. det. func.



i.e.

$$\text{DET}(\sigma_1, \dots, \sigma_n; \mathcal{P}) \simeq \text{Pic}(\mathcal{V}\sigma_1, \dots, \mathcal{V}\sigma_n; \mathcal{P})$$

equiv. of 2-cat

Can  $\mathcal{V}\sigma$  has a braided unital  $\mathcal{V}\sigma \times \mathcal{V}\sigma \xrightarrow{m} \mathcal{V}\sigma$

$\mathcal{V}\sigma$  is the Picard groupoid

$\Rightarrow k_0 = \pi_0 \mathcal{V}\sigma$  is a ring

$k_1 = \pi_1 \mathcal{V}\sigma$  is a  $\pi_1$ -bimodule