Determinant functors and the K-theory of tensor triangulated categories

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Background:

nice category (Waldhausen) W

$\downarrow$

~~K-theory spectrum~~

$\downarrow$

compute K_0 and K_1

associated $\underbrace{\text{Picard groupoid}}$ P — symmetric monoidal cat. + all morphisms are invertible

$K_0(W) = \pi_0 P$

$K_1(W) = \pi_1 P$

one way to get P: universal determinant functor

$\det : W \to P$

will work for $\mathcal{T}$

Triangulated Category $\mathcal{T}$

- additive category
- equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ (shift)
- class of diagrams called distinguished triangle

Δ: $X \to Y \to Z \to \Sigma X$ $\sim$

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, "+1"] & \end{tikzcd}$$

+ axioms

ex derived categories of abelian cat

Universal determinant functor on $\mathcal{T}$

$$d : \mathcal{T} \to V\mathcal{T}$$

triangulated cat — picard groupoid

det functor

- a functor $\mathrm{isom}(\mathcal{T}) \to V\mathcal{T}$
- for any distinguished triangle $x \to y \to z \to \Sigma x$ (Δ)

  $d(\Delta) : d(z) + d(x) \to d(y)$

  subject to some compatibily

universal

-
$$\begin{tikzcd} \mathcal{T} \arrow[r, "\text{det functor}"] \arrow[d, "d"'] & P \\ V\mathcal{T} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

  $\exists$

Tensor Triangulated Category

- triangulated category
- tensor category $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$

  unital, associate, +
- compatibility between the 2 structure

  e.g.

$$\begin{tikzcd} \mathcal{T} \times \mathcal{T} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \mathcal{T} \end{tikzcd} \qquad \begin{array}{c} \Sigma(x \otimes y) \\ \uparrow \cong \\ (\Sigma x) \otimes y \end{array}$$

→ can be viewed as a bunch of

$$\begin{tikzcd} \mathcal{T} \times \cdots \times \mathcal{T} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \mathcal{T} \arrow[r, "\text{det}"] & P \end{tikzcd}$$

i.e. info is encoded in the multi-cat. of triangulated cat. (2-cat)

Our extension: Multi-determinant functor

$$d : \mathcal{T}_1 \times \cdots \times \mathcal{T}_n \to P$$

- functor $d : \mathrm{isom}(\mathcal{T}_1) \times \cdots \times \mathrm{isom}(\mathcal{T}_n) \to P$
- determinant in each variable with pairwise compatibily

$$\begin{tikzcd} \mathcal{T} \times \cdots \times \mathcal{T} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & P \end{tikzcd}$$

multi-det

Take $P = V\mathcal{T}$

Theorem (and A.)

$$\begin{tikzcd}
\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \arrow[r, "\text{multi-det}"] \arrow[d, "d\times\cdots\times d"'] \arrow[dr, phantom, "\Uparrow \exists"] & P \\
V\mathcal{T}_1 \times \cdots \times V\mathcal{T}_n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

d is the universal det functor $\mathcal{T}_i \to V\mathcal{T}_i$

|| from un. det. func.

i.e.

$$\mathrm{DET}(\mathcal{T}_1, \dots, \mathcal{T}_n; P) \simeq \mathrm{Pic}(V\mathcal{T}_1, \dots, V\mathcal{T}_n; P)$$

equiv. of 2-cat

Can $\underbrace{V\mathcal{T}}_{\text{Picard groupoid}}$ have a biexact unital $V\mathcal{T} \times V\mathcal{T} \xrightarrow{m} V\mathcal{T}$

$\Rightarrow$ $K_0 = \pi_0 V\mathcal{T}$ is a ring

$K_1 = \pi_1 V\mathcal{T}$ is a π_1-bimodule