Traced Monads and Hopf Monads

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A bit of background story about this research

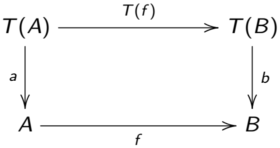

- In a few years we'll say: "You'll always remember where you were when the world stopped because of the COVID-19 pandemic in March 2020".
- Well I was at Kyoto University working with Hassei [1] on this very project!
- Not only that! The day the NBA/NHL postponed their season in March 2020 was the same day Hassei and I finally cracked the main result of this talk!
- Unfortunately I had to leave a few days later...

[1] If you google pictures of Masahito Hasegawa, you will find more pictures of Phil Scott and Peter Selinger than Hassei!

Lifting Structure to Eilenberg-Moore Categories

Let (T, μ, η) be a monad a category $\mathbb{X}$.

- Let $\mathbb{X}^T$ be its Eilenberg-Moore category, the category whose objects are T-algebras $(A, T(A) \xrightarrow{a} A)$ and whose maps are T-algebra morphisms $f : (A, a) \to (B, b)$:



  With forgetful functor $U : \mathbb{X}^T \to \mathbb{X}$.

- A structure on $\mathbb{X}$ is lifted to $\mathbb{X}^T$ if $\mathbb{X}^T$ also has said structure and the forgetful functor U preserves said structure strictly.

- In most cases, lifting structure can be characterized on the monad T itself without mentioning its T-algebras. This is often done by requiring extra structure on the monad (like natural transformations, distributive laws, etc.)

TODAY'S STORY: Lifting Traced Monoidal Structure

Lifting Trace

- **Traced monoidal categories** are symmetric monoidal categories which have a **trace** operator, which generalizes the notion of traces of matrices and the notion of feedback/loops.

  André Joyal, Ross Street, Dominic Verity (1996). **Traced monoidal categories**.

- A **traced monad** is a monad T on a traced monoidal category $\mathbb{X}$ which lifts the traced monoidal structure to $\mathbb{X}^T$. Explicitly:

  1. The Eilenberg-Moore category $\mathbb{X}^T$ is a traced monoidal category
  2. The forgetful functor $U : \mathbb{X}^T \to \mathbb{X}$ preserves the traced monoidal structure strictly.

- An open question has been trying to provide a characterization of traced monads without explicitly mentioning the Eilenberg-Moore category.

Lifting Compact Closed Structure

- Traced monoidal categories are closely linked to **compact closed categories**.
  - Every compact closed category is a traced monoidal category;
  - Every traced monoidal category embeds into a compact closed category.
- Monads which lift compact closed structure are called **Hopf monads**.
- Hopf monads can be described on arbitrary monoidal categories without mentioning the Eilenberg-Moore category.
- A natural question to ask is what is the relationship between Hopf monads and traced monads. Can Hopf monads be used to help characterize traced monads?

Summary of Today

- We introduce the notion of **trace-coherence** for Hopf monads, which can be stated without mentioning the Eilenberg-Moore category.
- **Main Result:** A Hopf monad is traced if and only if it is trace-coherent.
- Not every traced monad is a Hopf monad.
- Not every Hopf monad is a traced monad ($\leftarrow$ counterexample was not easy to find!)

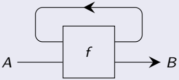
Traced Monoidal Categories

For a symmetric monoidal category $\mathbb{X}$, we use $\otimes$ for the tensor product and I for the unit.

Definition

A **traced monoidal category** is a symmetric monoidal category $\mathbb{X}$: equipped with a **trace** Tr, which is a family of operators (indexed by triples of objects $X, A, B \in \mathbb{X}$):

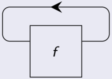
$$\mathrm{Tr}^X_{A,B} : \mathbb{X}(X \otimes A, X \otimes B) \to \mathbb{X}(A, B) \qquad \frac{f : X \otimes A \to X \otimes B}{\mathrm{Tr}^X_{A,B}(f) : A \to B}$$



satisfying axioms which generalize the trace operator for matrices (like the cyclic property).

André Joyal, Ross Street, Dominic Verity (1996). **Traced monoidal categories**.

For an endomorphism $A \xrightarrow{f} A$, its trace $I \xrightarrow{\mathrm{Tr}(f) = \mathrm{Tr}^A_{I,I}(f)} I$ is a loop:



Examples: Compact Closed Categories

Proposition

A **compact closed category** *is a symmetric monoidal category such each object A has a dual A^*, with maps $A \otimes A^* \xrightarrow{\cup} I$ and $I \xrightarrow{\cap} A^* \otimes A$ that satisfy the* **snake equations**. *Every compact closed category is a traced monoidal category where:*

$$\mathrm{Tr}^X_{A,B}(f) := \; A \cong I \otimes A \xrightarrow{\cap \otimes 1} X^* \otimes X \otimes A \xrightarrow{1 \otimes f} X^* \otimes X \otimes B \cong X \otimes X^* \otimes B \xrightarrow{\cup \otimes 1} I \otimes B \cong B$$

Example

- For any commutative ring R, $\mathrm{MAT}(R)$ is compact closed and the trace operator corresponds to partial trace of matrices. In particular for a $n \times n$ square matrix A, which is an endomorphism $n \xrightarrow{A} n$, its trace $1 \xrightarrow{\mathrm{Tr}(A)} 1$ is precisely the sum of its diagonal coordinates.

- REL with $\otimes = \times$ is compact closed and the resulting trace is defined as follows:

$$\frac{R \subseteq (X \times A) \times (X \times B)}{\mathrm{Tr}^X_{A,B}(R) = \{(a,b) |\; \exists x \in X. \, ((x,a),(x,b)) \in R\} \subseteq A \times B}$$

- Every traced monoical category $\mathbb{X}$ embeds into a compact closed category $\mathrm{INT}(\mathbb{X})$, and the embedding $J : \mathbb{X} \to \mathrm{INT}(\mathbb{X})$ preserves the traced monoidal structure strictly.

Non-Compact Closed Examples

NOT EVERY TRACED MONOIDAL CATEGORY IS COMPACT CLOSED!

Example

- The poset $(\mathbb{N}, \leq)$ is a traced monoidal category where $\otimes = +$ and the trace is:

$$\frac{k + n \leq k + m}{n \leq m}$$

- Every unique decomposition category is a traced monoidal category:

  Esfandir Haghverdi (2000). **Unique decomposition categories, Geometry of Interaction and combinatory logic**.

  The idea here is that $X \otimes A \xrightarrow{F} X \otimes B$ can be decomposed into four maps:

$$X \xrightarrow{f} X \qquad X \xrightarrow{g} B \qquad A \xrightarrow{h} X \qquad A \xrightarrow{k} B$$

  so its trace $A \xrightarrow{\mathrm{Tr}^X_{A,B}(F)} B$ is defined as an infinite sum:

$$\mathrm{Tr}^X_{A,B}(F) = k + \sum_{n \in \mathbb{N}} g \circ f^n \circ h$$

  - REL with $\otimes = \sqcup$ (disjoint union)
  - PAR with $\otimes = \sqcup$ (disjoint union)

Traced Monads

Traced monads are monads which lift traced monoidal structure:

1. How do we lift the symmetric monoidal structure?
2. How do we lift the trace?

Comonoidal Monads

Definition

A **symmetric comonoidal monad** on a symmetric monoidal category $\mathbb{X}$ is a monad (T, μ, η) equipped with:

$$T(A \otimes B) \xrightarrow{m_{A,B}} T(A) \otimes T(B) \qquad\qquad T(I) \xrightarrow{m_I} I$$

which makes T a (lax) comonoidal functor (i.e. coherences with the associativity, symmetry, unit isomorphisms) and μ and η comonoidal natural transformations.

Proposition (Moerdijk)

A monad (T, μ, η) is a symmetric comonoidal monad if and only if $\mathbb{X}^T$ is a symmetric monoidal category such that the forgetful functor preserves the symmetry monoidal structure strictly (i.e. a strict monoidal functor).

The Eilenberg-Moore category of a symmetric comonoidal monad is a symmetric monoidal category:

$$(A, a) \otimes (B, b) = (A \otimes B, T(A \otimes B) \xrightarrow{m_{A,B}} T(A) \otimes T(B) \xrightarrow{a \otimes b} A \otimes B) \qquad (I, T(I) \xrightarrow{m_I} I)$$

I. Moerdijk. (2002). **Monads on tensor categories**.

Traced Monads

Definition

A **traced monad** on a traced monoidal category $\mathbb{X}$ is a symmetric comonoidal monad (T, μ, η, m, m_I) such that $\mathbb{X}^T$ is a traced monoidal category and the forgetful functor preserves the traced monoidal structure strictly.

In other words, the trace of T-algebra map is again a T-algebra map:

$$\frac{(X, x) \otimes (A, a) \xrightarrow{f} (X, x) \otimes (B, b)}{(A, a) \xrightarrow{\mathrm{Tr}^X_{A,B}(f)} (B, b)}$$

$$\begin{array}{ccc} T(X \otimes A) & \xrightarrow{T(f)} & T(X \otimes B) \\ \downarrow{\scriptstyle m_{X,A}} & & \downarrow{\scriptstyle m_{X,B}} \\ T(X) \otimes T(A) & & T(X) \otimes T(B) \\ \downarrow{\scriptstyle x \otimes a} & & \downarrow{\scriptstyle x \otimes b} \\ X \otimes A & \xrightarrow[f]{} & X \otimes B \end{array} \quad \Rightarrow \quad \begin{array}{ccc} T(A) & \xrightarrow{T\left(\mathrm{Tr}^X_{A,B}(f)\right)} & T(B) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[\mathrm{Tr}^X_{A,B}(f)]{} & B \end{array}$$

Problem: This is a bit disappointing because we mention the T-algebras...

Hopf Monads

Definition

A **symmetric Hopf monad** on a symmetric monoidal category $\mathbb{X}$ is a symmetric comonoidal monad (T, μ, η, m, m_I) such that the fusion operator:

$$h_{A,B} := \ T(T(A) \otimes B) \xrightarrow{m_{T(A),B}} TT(A) \otimes T(B) \xrightarrow{\mu_A \otimes 1_{T(B)}} T(A) \otimes T(B)$$

is a natural isomorphisms, so $T(A) \otimes T(B) \cong T(T(A) \otimes B)$.

A. Bruguieres, S. Lack, and A. Virelizier. (2011). **Hopf monads on monoidal categories**.

Proposition (Bruguieres, Lack, Virelizier)

If $\mathbb{X}$ is compact closed[a]*, then a monad (T, μ, η) is a Hopf monad if and only if $\mathbb{X}^T$ is a compact closed category such that the forgetful functor preserves the compact closed structure strictly.*

[a] In fact, this is true for symmetric closed categories!

Examples of Hopf Monads

Example

In any symmetric monoidal category, for an object H, $T = H \otimes -$ if a symmetric Hopf monad if and only if H is a commutative Hopf algebra. The monad is obtain from the algebra structure:

$$H \otimes H \xrightarrow{\nabla} H \qquad\qquad I \xrightarrow{u} H$$

The comonoidal structure is obtained from the coalgebra structure:

$$H \xrightarrow{\Delta} H \otimes H \qquad\qquad H \xrightarrow{e} I$$

while the invertibility of the fusion operators comes from the antipode:

$$H \xrightarrow{S} H$$

These are called **representable Hopf monads**.

While for any Hopf monad, $T(I)$ is a Hopf algebra, **not every Hopf monad is representable!**

Example

For $(\mathbb{N}, \leq)$, define: $T(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

Then T is a symmetric Hopf monad since $T(n) + T(m) = T(T(n) + m)$.

When is a Hopf Monad a Traced Monad?

Definition

A symmetric Hopf monad on a traced monoidal category is **trace-coherent** if for any map:

$$T(X) \otimes A \xrightarrow{f} T(X) \otimes B$$

the image of its trace:

$$A \xrightarrow{\mathrm{Tr}^{T(X)}_{A,B}(f)} B$$

under T:

$$T(A) \xrightarrow{T\left(\mathrm{Tr}^{T(X)}_{A,B}(f)\right)} T(B)$$

is equal to taking the trace of the composite:

$$T(X) \otimes T(A) \xrightarrow{h^{-1}_{X,A}} T(T(X) \otimes A) \xrightarrow{T(f)} T(T(X) \otimes B) \xrightarrow{h_{X,B}} T(X) \otimes T(B)$$

That is:

$$T\left(\mathrm{Tr}^{T(X)}_{A,B}(f)\right) = \mathrm{Tr}^{T(X)}_{T(A),T(B)}\left(h_{X,B} \circ T(f) \circ h^{-1}_{X,A}\right)$$

Proposition (Hasegawa, Lemay)

A symmetric Hopf monad is a traced monad if and only if it is trace-coherent.

What kinds of Hopf monads are trace-coherent?

Proposition

For any traced monoidal category, every representable symmetric Hopf monad $T = H \otimes -$ is trace-coherent, and therefore a traced monad. The Eilenberg-Moore category in this case is the category of H-modules, which will be a traced monoidal category.

Proposition

Every symmetric Hopf monad on a compact closed category is trace-coherent, and therefore a traced monad.

Proposition

For any traced monoidal category, every **idempotent** *symmetric Hopf monad (i.e. $TT \cong T$) is trace-coherent, and therefore a traced monad. Furthermore, if the unit I is an initial object, then a symmetric Hopf monad is trace-coherent if and only if it is idempotent. In particular, this covers examples where $\otimes$ is a coproduct (like* REL *or* PAR*).*

Not all traced monads are Hopf monads

If $\mathbb{X}$ is a traced monoidal category, note that $\mathbb{X}^{op}$ is also.

Example

Suppose that $\mathbb{X}$ is a traced symmetric monoidal closed category with internal hom $A \multimap B$.

- Consider the compact closed category $\mathsf{INT}(\mathbb{X}^{op})$. Briefly, the objects are pairs (X, A)
- Define the monad T on objects as follows: $T(X, A) = (A \multimap X, I)$.
- The Eilenberg-Moore category $\mathsf{INT}(\mathbb{X}^{op})^T \cong \mathbb{X}^{op}$, and so the forgetful functor can be interpreted as the embedding $J : \mathbb{X}^{op} \to \mathsf{INT}(\mathbb{X}^{op})$, so $J(A) = (A, I)$.
- By construction, the embedding $J : \mathbb{X}^{op} \to \mathsf{INT}(\mathbb{X}^{op})$ preserves the traced monoidal structure strictly. Therefore, T is a traced monad.
- However, T is in general not a Hopf monad since:

$$T(T(X, A) \otimes (Y, B)) = (B \multimap ((A \multimap X) \otimes Y), I)$$
$$\neq ((A \multimap X) \otimes (B \multimap Y), I) = T(X, A) \otimes T(Y, B)$$

Not all Hopf monads are traced monads

The trace operator is a **structure** and not a property. In other words, a symmetric monoidal category $\mathbb{X}$ can have two different trace operators for the same tensor product $\otimes$.

Example

Let $\mathbb{X}$ be a symmetric monoidal category with two distinct trace operator Tr and $\overline{\mathsf{Tr}}$. Suppose that $\mathbb{X}$ also has distributive biproducts $\oplus$:

$$(X \oplus Y) \otimes (A \oplus B) \cong (X \otimes A) \oplus (X \otimes B) \oplus (Y \otimes A) \oplus (Y \otimes B)$$

Now $\mathbb{X} \times \mathbb{X}$ is again a traced monoidal category where: $\mathsf{Tr}(-,-) = (\mathsf{Tr}(-), \overline{\mathsf{Tr}}(-))$

Now define the monad T on objects as follows $T(X,Y) = (X \oplus Y, X \oplus Y)$

That T is a Hopf monad follows from distributivity between $\oplus$ and $\otimes$.

The Eilenberg-Moore category $(\mathbb{X} \times \mathbb{X})^T \cong \mathbb{X}$, and so the forgetful functor can be interpreted as the diagonal functor $\Delta : \mathbb{X} \to \mathbb{X} \times \mathbb{X}$. But with either of the traces of $\mathbb{X}$ we see that:

$$\Delta(\mathsf{Tr}(-)) = (\mathsf{Tr}(-), \mathsf{Tr}(-)) \neq (\mathsf{Tr}(-), \overline{\mathsf{Tr}}(-)) = \mathsf{Tr}(-,-)$$

$$\Delta(\overline{\mathsf{Tr}}(-)) = (\overline{\mathsf{Tr}}(-), \overline{\mathsf{Tr}}(-)) \neq (\mathsf{Tr}(-), \overline{\mathsf{Tr}}(-)) = \mathsf{Tr}(-,-)$$

So the forgetful/diagonal functor does not preserve the trace! Therefore T is not a traced monad.

Some final thoughts...

- We were able to characterize when Hopf monads lift traced monoidal structure without mentioning the algebras.
- We are able to get a trace-coherent condition for slightly more general monads:

$$T(T(A) \otimes T(B)) \cong TT(A) \otimes T(B)$$

- Regarding traced monads $\rightarrow$ Hopf monads: can the trace be sometimes used to build inverses for the fusion operators? (It doesn't seem like it... but maybe!)
- What about Hopf monads on unique decomposition categories?
- How does this story interact with the INT construction?
- Still working on getting a characterization of traced monads without mentioning the algebras!

Thank You!

HOPE YOU ENJOYED MY TALK!
THANK YOU FOR LISTENING!
MERCI!