SHEAF THEORY, PART II

We'll follow Tennison.

Let X be a top space. A <u>presheaf</u> of sets on X is a functor

$$\mathcal{O}(X)^{op} \to \mathrm{Set}$$

Equivalently,

- For each open set $U \in \mathcal{O}(X)$, a set $F(U)$
- $\forall\, U, V \in \mathcal{O}(X)$ with $V \subseteq U$

  a map $\rho^U_V : F(U) \to F(V)$

  such that

  - $\forall\, U$ $\rho^U_U = \mathrm{id}_{F(U)}$
  - If $W \subseteq V \subseteq U$ in $\mathcal{O}(X)$

  $$\rho^U_W = \rho^V_W \circ \rho^U_V$$

- $F(U)$ is called the set of <u>sections</u> of F over U
- The maps ρ^U_V are called <u>restriction</u> maps

A presheaf of abelian groups is a presheaf of sets s.t. $\forall U$ $F(U)$ is an abelian group & for all U, V ρ^U_V is a group homomorphism

Similarly for rings, groups, vector spaces, etc

Def'n: Let ~~F~~ F be a presheaf on a space X.

F is separated if

- Whenever $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $U \in \mathcal{O}(X)$ and s, s' satisfy that $s, s' \in F(U)$ and $\forall \lambda$ $\rho^U_{U_\lambda}(s) = \rho^U_{U_\lambda}(s')$

then $s = s'$. Tennison calls this condition (M)

- Condition (G) says

Suppose $\{U_\lambda\}_{\lambda \in \Lambda}$ is a cover of U

and suppose we have a family of sections

$\{ s_\lambda \in U_\lambda \}_{\lambda \in \Lambda}$ such that

$$\forall \lambda, \mu \in \Lambda \quad \rho^{U_\lambda}_{U_\lambda \cap U_\mu}(s_\lambda) = \rho^{U_\mu}_{U_\lambda \cap U_\mu}(s_\mu)$$

then there is an $s \in F(U)$ s.t. $\forall \lambda$

$$\rho^{U}_{U_\lambda}(s) = s_\lambda.$$

Note: If F satisfies condition (m), then the $s \in F(U)$ above is unique.

<u>Key example</u> Let E be a space and $p : E \to X$ a continuous map. Define

$$F : \mathcal{O}(X)^{op} \to \mathrm{Set}$$

by $F(U) = \{ \sigma : U \to E$ continuous s.t.

$p \circ \sigma = \mathrm{id}_U$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ U \arrow[ur, "\sigma"] \arrow[r, hook] & X \end{tikzcd}$$

Be sure to understand this example.

Note that nothing we have said so far really used any category theory.

Let's restate the definition of sheaf in terms of equalizers.

Def'n: Let $\mathcal{C}$ be any category and suppose I have

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

a pair of parallel arrows in $\mathcal{C}$.

The equalizer of f and g, if it exists, is an object E, with a map $e: E \to X$ s.t.

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

$f \circ e = g \circ e$ and if F is any object with a map $F \xrightarrow{h} X$ s.t. $f \circ h = g \circ h$,

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ F \arrow[ur, "h"'] & & \end{tikzcd}$$

then there is a unique map $q: F \to E$
such that $e \circ q = h$

$$\begin{tikzcd} E \arrow[r, "e"] & X \\ F \arrow[u, dotted, "q"] \arrow[ur, "h"'] & \end{tikzcd}$$

As always, equalizers are unique up to isomorphism.

In Set $E = \{x \in X \mid f(x) = g(x)\}$

We're going to set up a diagram as follow,
Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be any open covering of any open set $U \in \mathcal{O}(X)$

Will define a map

$$a: F(U) \longrightarrow \prod_{\lambda \in \Lambda} F(U_\lambda)$$

This is an infinite product. We'll define a by saying what the image of a is in each component.

If $s \in F(U)$, then the λ-component of $a(s)$ will be denoted $a(s)_\lambda$. It's defined by

$$a(s)_\lambda = \underbrace{\rho^U_{U_\lambda}(s)}$$

By definition, this is in $F(U_\lambda)$

Since this is defined for arbitrary λ, we have constructed a map

$$F(U) \xrightarrow{a} \prod_{\lambda \in \Lambda} F(U_\lambda)$$

Now I want to build

$$\prod_{\lambda \in \Lambda} F(U_\lambda) \overset{b}{\underset{c}{\rightrightarrows}} \prod_{(\lambda,\mu) \in \Lambda \times \Lambda} F(U_\lambda \cap U_\mu)$$

The domain of these 2 maps is a product.

Let $(s_\lambda)_{\lambda \in \Lambda}$ be an element of $\prod_{\lambda \in \Lambda} F(U_\lambda)$

Define

$$b\left((s_\lambda)_{\lambda\in\Lambda}\right) = \left(\rho^{U_\lambda}_{U_\lambda\cap U_\mu}(s_\lambda)\right)_{(\lambda,\mu)\in\Lambda\times\Lambda}$$

$$c\left((s_\lambda)_{\lambda\in\Lambda}\right) = \left(\rho^{U_\mu}_{U_\lambda\cap U_\mu}(s_\mu)\right)_{(\lambda,\mu)\in\Lambda\times\Lambda}$$

Thm: A presheaf on $(X, \mathcal{O}(X))$ is a sheaf if and only if the diagram

$$F(U) \xrightarrow{a} \prod_{\lambda\in\Lambda} F(U_\lambda) \underset{c}{\overset{b}{\rightrightarrows}} \prod_{(\lambda,\mu)\in\Lambda\times\Lambda} F(U_\lambda\cap U_\mu)$$

is an equalizer diagram for all opens U and all open covers of U, $\{U_\lambda\}_{\lambda\in\Lambda}$

Proof: You need to stare at this awhile. But then you'll realize that condition (G) says that $b\circ a = c\circ a$ and condition (M) ~~says that~~ gives uniqueness. $\square$

Direct Limits (p 3 of Tennison)

Special case of colimits. We just need this level of generality.

A directed set Λ is a set with a preorder ($x \le x$, $x \le y, y \le z \Rightarrow x \le z$, but we don't assume $x \le y$ & $y \le x \Rightarrow x = y$)

$\boxed{\text{In the present case, this doesn't matter}}$

such that

$$* \quad \forall\, \alpha, \beta \in \Lambda \ \exists\, \gamma \in \Lambda \text{ s.t. } \alpha \le \gamma \text{ and } \beta \le \gamma.$$

Note we are not assuming a least upper bound, just some upper bound

A direct system of sets indexed by a directed set Λ is a family of sets indexed by Λ

$$\{U_\alpha\}_{\alpha \in \Lambda}$$

together with

$\forall\, \alpha, \beta$ with $\alpha \leq \beta$ a map

$$\rho_{\alpha\beta} : U_\alpha \to U_\beta$$

s.t. 1) $\rho_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$

2) if $\alpha \leq \beta \leq \gamma$, then

$$\rho_{\alpha\gamma} = \rho_{\beta\gamma} \circ \rho_{\alpha\beta}$$

Consider $\mathcal{O}(X)$ and say $U \leq V$ iff $U \supseteq V$ and given a presheaf on X, we get a directed system for $\Lambda = \mathcal{O}(X)$ with $\{F(U)\}_{U \in \mathcal{O}(X)}$

$$\rho_{UV} = \rho^U_V$$

Directed System

$$\begin{tikzcd}
U_\alpha \arrow[dr] & & U_\delta \arrow[dr] & \\
& U_\gamma & & {} \\
U_\beta \arrow[ur] & & U_\varepsilon \arrow[ur] &
\end{tikzcd}$$

We wish to construct a colimit of ~~this system~~ directed systems.

Def'n: Given a directed system as above, a target for the system is a set V and a collection of maps

$$\{\sigma_\alpha : U_\alpha \to V\}_{\alpha \in \Lambda}$$

Note: we need a map for every $\alpha \in V$ s.t. if $\alpha \leq \beta$

$$\begin{tikzcd} U_\alpha \arrow[d, "\rho_{\alpha\beta}"'] \arrow[dr, "\sigma_\alpha"] & \\ U_\beta \arrow[r, "\sigma_\beta"'] & V \end{tikzcd}$$

Then, a direct limit for the system is a target U which is universal in the following sense: So $(U, \tau_\alpha : U_\alpha \to U)$

For any target V $\exists$ a <u>unique</u> map

$f: U \to V$ s.t. $\forall \alpha \in \Lambda$

$$\begin{tikzcd} U_\alpha \arrow[r, "\sigma_\alpha"] \arrow[dr, "\tau_\alpha"'] & V \\ & U \arrow[u, "f"'] \end{tikzcd}$$

commutes.

Lemma: Any two direct limits for a direct system are isomorphic in a way compatible with all the τ_α maps

Thm: Direct limits of direct systems exist.

Proof: We know that the category Set of sets and functions is cocomplete.

But it's preferable to give a direct construction

This is on p6 of Tennison.

Suppose I have a direct system

$\{U_\alpha\}_{\alpha \in \Lambda}$ and $\{\rho_{\alpha\beta} \mid \alpha \leq \beta\}$

start by taking the disjoint union

$$W = \coprod_{\alpha \in \Lambda} U_\alpha$$

We'll define an equivalence relation by

$$u \in U_\alpha \sim v \in U_\beta$$

if $\exists\, \gamma$ with $\alpha, \beta \leq \gamma$ and

$$\rho_{\alpha\gamma}(u) = \rho_{\beta\gamma}(v)$$

Intuition: u, v eventually become equal.

Then $U = W/_\sim$ is a direct limit.

I need maps $\tau_\alpha : U_\alpha \to U$. It's given by

quotient map

$$U_\alpha \to W \longrightarrow W/_\sim = U$$

inclusion to a component

This works. Denote it by $\varinjlim F(U)$

Tennison also discusses direct systems of abelian groups. Construction is the same.

Important in sheaf cohomology

Let F be a presheaf. consider $x \in X$.

Consider the sets $\{F(U) \mid x \in U\}$. This is still a directed set, using ρ^U_V.

Def'n: The limit of this directed set is denoted F_x and is called the stalk at X

$$F_x = \varinjlim_{x \in U} F(U)$$

So we have maps $F(U) \to F_x$

$$s \mapsto s_x = [s]$$

↑ equivalence class of F

The elements of F_x are called the <u>germs</u> of F at x

<u>Ex 1</u> 1) If F is a constant presheaf $F(U) = A$ $\forall U \in \mathcal{O}(X)^{op}$, then $F_x = A$ for all $x \in X$.

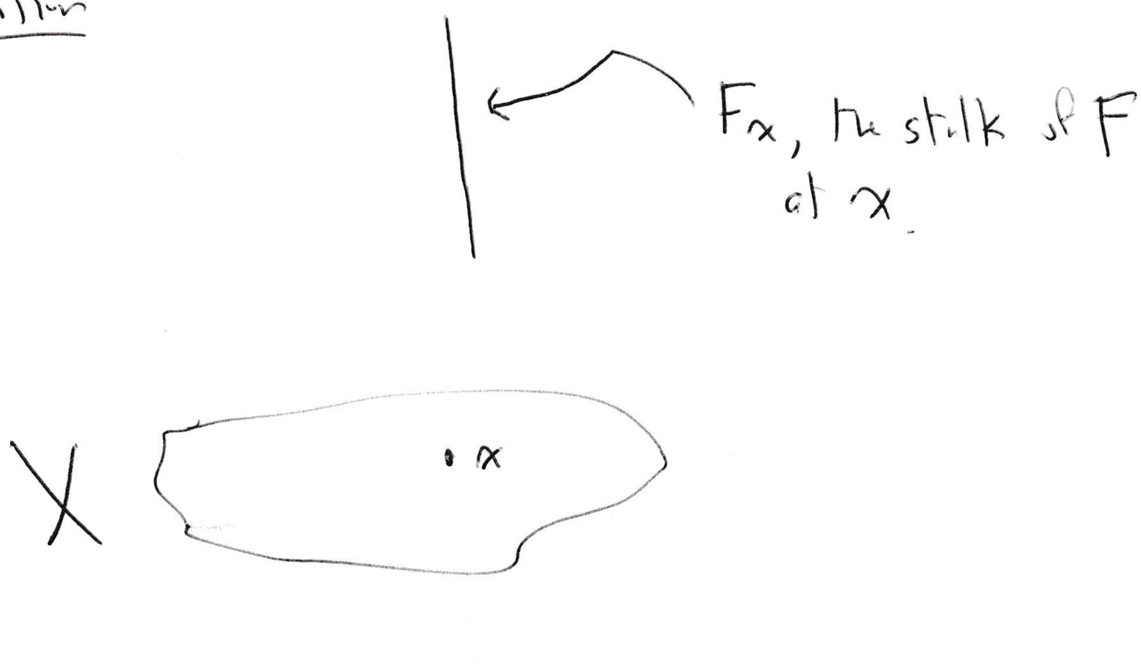
2) Let Y be another topological space

Let $C^Y(U) =$ set of continuous maps $U \to Y$

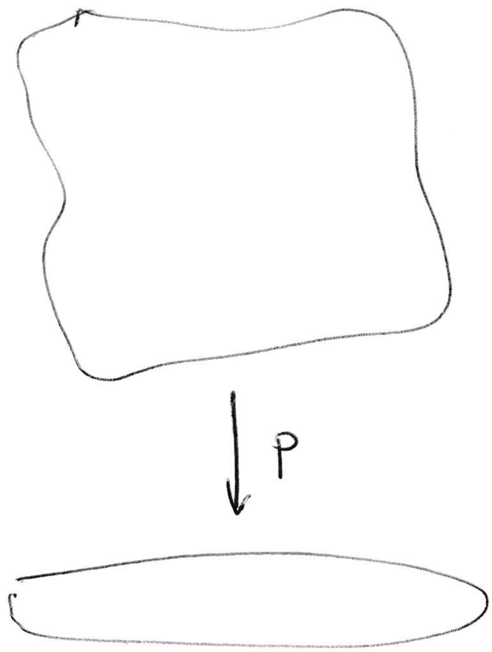
Then ρ^U_V is just the usual restriction map.

A germ at x is a function defined on some neighborhood of x and two germs are equal if they agree on some neighborhood of x

<u>Intuition</u>



Eventually, we'll combine the stalks into a space, sitting over X.



p will be a particularly well-behaved continuous map. (a ~~local~~ local homeomorphism)

Some technical lemmas

Recall that a morphism of sheaves or presheaves is just a natural transformation.

Lemma: If F is a presheaf satisfying property (M) and G is an arbitrary presheaf and

$$\sigma, \tau : F \to G$$

such that $\forall x \in X$

$$\sigma_x = \tau_x$$

i.e. they agree on stalks, then $\sigma = \tau$.

(σ_x is the restriction of σ to the stalk at x

$\sigma_x : F_x \to G_x$)