Lecture

Def'n: A topos is a category $\mathcal{C}$ which has

- finite limits and colimits
- cartesian closed
- Subobject classifier

This last means there is an object Ω such that $\forall$ objects A

$$\mathrm{Sub}(A) \cong \mathrm{Hom}(A, \Omega)$$

and $\mathrm{Sub}(A)$ is the set of subobjects of A.

A subobject of A is an equivalence class of monies into A

$$\begin{tikzcd}
B \arrow[dr, hook] \arrow[d, phantom, "\cong" rotate=90] & \\
B' \arrow[r, hook] & A
\end{tikzcd}$$

Equivalence means there is an isomorphism between B and B' making the above triangle commute.

This definition is highly redundant.

- The existence of finite colimits follows from the rest of the structure.
  - We only need exponentials of the form
  - Ω^A to prove the existence of all exponentials
- Note Ω^A is denoted $\mathcal{P}(A)$ the power object of A.
- Ω is the object of truth values of the topos.

In this interpretation, the topos is a universe in which one can do mathematics. Ω need not be a Boolean algebra, but is always a Heyting algebra. So we have models of intuitionistic set theory.

Ex: Sets, G-Sets, when G is a group.
FiniteSets, $Set^{\mathcal{C}^{op}}$: category of presheaves on $\mathcal{C}$.

We'll just prove that $Set^{\mathcal{C}^{op}}$ is cartesian closed.

Products are straightforward

$$F \times G(c) = F(c) \times G(c)$$

The terminal object is the constant functor, sending everything to $\{*\}$.

We need to construct F^G. Note that it cannot be

$$F^G(c) = F(c)^{G(c)} = \mathrm{Hom}_{Set}((G(c), F(c)))$$

Since this can't be made functorial in C.

We need

$$\mathrm{Hom}(F \times G, H) \cong \mathrm{Hom}(F, H^G)$$

I'm going to write this as

$$\mathrm{Nat}(F \times G, H) \cong \mathrm{Nat}(F, H^G) \quad *$$

since F, G, H are functors $\mathcal{C}^{op} \to Set$.

Let's suppose F is representable, i.e.

$$F = \mathrm{Hom}(-, C) \text{ for some } C \in |\mathcal{C}|$$

We'll use Yoneda notation and denote $\mathrm{Hom}(-, c)$ as $y(c)$. So we get

$$\mathrm{Nat}(y(c) \times Q, R) \cong \mathrm{Nat}(y(c), R^Q)$$

And here's the trick

$$\mathrm{Nat}(y(c), R^Q) \cong R^Q(c) \quad **$$

By Yoneda lemma.

<u>Reminder on Yoneda lemma.</u>

If $F : \mathcal{C}^{op} \to \mathrm{Set}$, $\mathrm{Nat}(y(c), F) \cong F(c)$

So the point is, we are trying to define $R^Q(c)$. But $**$ gives us a formula for it.

So let's define

$$R^Q(c) = \mathrm{Nat}(y(c) \times Q, R)$$

and this works.

Just check details.

Sheaf Theory

The notion of presheaf originated over topological spaces. If $(X, \mathcal{O}(X))$ is a topological space, then $\mathcal{O}(X)$ as a poset determines a category, and the original examples were functions

$$\mathcal{O}(X)^{op} \to Set$$

Then sheaves were especially nice presheaves.

Note: It doesn't make sense to talk about a sheaf on an arbitrary category, unless there is some additional structure, a Grothendieck topology.

References

Tennison - Sheaf Theory

Bredon - Sheaf Theory (electronic copy available)

Article I will email.

Let $(X, \mathcal{O}(X))$ be a topological space.

A presheaf (of sets) over X is a functor

$$F : \mathcal{O}(X)^{op} \to Set$$

poset viewed as a category.

So for each $U \subseteq X$ open, I have a set $F(U)$.

For each U, V open with $U \subseteq V$ a function

$$r_{U,V} : F(V) \to F(U)$$

Note reversal of direction
Also note some books write this $r_{V,U}$

These must satisfy 1) $r_{U,U} = id_{F(U)}$

2) Given $U \subseteq V \subseteq W$

$$r_{W,U} = r_{V,U} \circ r_{W,V}$$

These 2 conditions are precisely functoriality.

<u>Key examples</u>:

1) For each U, let $F(U)$ be the continuous real-valued functions on U.

If $U \subseteq V$ $r : F(V) \to F(U)$ is restriction.

$$f: V \to \mathbb{R} \mapsto f|_U$$

NB The restriction of a continuous function is continuous.

2) Constant presheaves.

<u>NOTE</u>: This won't be relevant to what we are doing, but you can also talk about presheaves (or sheaves) of rings, groups, etc.

If F is a presheaf of rings, I can talk about a presheaf of modules over F.

So for each U, I have an $F(U)$-module etc

<u>Sheaf Cohomology</u>

See Bredon, Iversen: Cohomology of Sheaves

Def'n: Let $F : \mathcal{O}(X)^{op} \to Set$ be a presheaf of sets.

1) F is <u>separated</u> when, if $U \subseteq X$ is open, and $\{U_\alpha\}_{\alpha \in A}$ is an open covering of U, i.e.

$$\bigcup_{\alpha \in A} U_\alpha = U$$

and suppose $s, t \in F(U)$ and

$$\forall \alpha \in A \quad r_{U,U_\alpha}(s) = r_{U,U_\alpha}(t),$$

then $s = t$.

Proposition 1.1.15: If S is a set and F is the presheaf on X defined by

$$F(U) = \{f : U \to S\}$$

then F is separated.

Why?

<u>Defn</u>: Suppose $U \subseteq X$ is open and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U. Suppose $\forall \alpha \in A$, we have $s_\alpha \in F(U_\alpha)$ and these elements satisfy

$$\forall \alpha_1, \alpha_2 \quad r_{U_{\alpha_1}, U_{\alpha_1} \cap U_{\alpha_2}}(s_{\alpha_1}) = r_{U_{\alpha_2}, U_{\alpha_1} \cap U_{\alpha_2}}(s_{\alpha_2})$$

Then $\exists\, s \in F(U)$ s.t.

$$\forall \alpha \in A \quad r_{U, U_\alpha}(s) = s_\alpha$$

Then F has the <u>gluing property</u>

Intuition: Think of our functions example where $F(U) =$ continuous functions $U \to \mathbb{R}$.

If I have a family of functions defined on the elements of an open cover of U and <u>they agree on the overlap</u>, Then I can glue them together to build ~~the~~ a continuous function on all of U.

Def'n: A presheaf on a space $(X, \mathcal{O}(X))$ is a sheaf if it is separated and satisfies the gluing condition.

Lemma: If F is a sheaf, then $F(\emptyset)$ is a 1-element set.

Examples of sheaves

1) $X = \{x\}$ and S is any set, define

$$F(\{x\}) = S$$
$$F(\emptyset) = \{*\}$$

2) Let X be any space and $x_0 \in X$

Define $F: \mathcal{O}(X)^{op} \to Set$ by

$$F(U) = \begin{cases} S & \text{if } x_0 \in U \\ \{*\} & \text{if not} \end{cases}$$

3) The presheaf of continuous real-valued functions is a sheaf.

4) Let X, Y be spaces & $f: Y \to X$ a continuous function. Define a presheaf on X by $\Gamma(U) = \{ s: U \to Y \mid f \circ s = id_U \}$

Γ is a sheaf. It is called the sheaf of sections on X. More generally,

Def'n: Given any presheaf, the elements of $F(U)$ are called sections over U and elements of $F(X)$ are called global sections.

Def'n: If F, G are sheaves on X, a morphism $a: F \to G$ is just a natural transformation. The resulting category is denoted $Sh(X)$.

Thm: $Sh(X)$ is a topos.

Looking ahead, how does one prove this?

Let Pre(X) denote the category of presheaves on X with natural transformations as morphisms.

We have

$$Sh(X) \subseteq Pre(X)$$

and in fact, there is a left adjoint to this inclusion, called <u>sheafification</u>.

$$SH : Pre(X) \to Sh(X).$$

One uses <u>sheafification</u> to transfer the topos structure of Pre(X) to Sh(X).

<u>Remark:</u> Evidently, one can talk about sheaves of rings, groups, modules, etc.

In algebraic geometry, one works with <u>ringed spaces</u> $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

Grothendieck topology

Grothendieck realized that he needed to generalize the notion of space. This led to huge contributions in Algebraic Geometry & Number Theory. Let $\mathcal{C}$ be a category.

A sieve on $c \in |\mathcal{C}|$ is a subfunctor of $\mathrm{Hom}(-, c)$, so it is a certain family of arrows with ~~the~~ codomain c.

For the case of a topological space, a sieve on a set U collects a set of open subsets of U.

A Grothendieck topology is a collection for each object $c \in |\mathcal{C}|$ of sieves, called the covering sieves. These must satisfy certain axioms.

These axioms are sufficient to define the notion of a sheaf. If we're working with a space X, we get the usual definition.

The category of sheaves on a category equipped with a Grothendieck topology is called a Grothendieck topos.

Verdier characterized which categories arise as Grothendieck toposes.