# Lecture

## Intro To Topos Theory

A topos is a lot of different things.

- a generalized space. A (Grothendieck) topos is an axiomatization of the category of sheaves on a space.
- a generalized category of sets and functions. A (Lawvere-Tierney) topos was viewed as a foundation for mathematics.

One can consider

- Axiom Of Choice
- Continuum Hypothesis

etc

- a generalized algebraic theory.
- a model of constructive mathematics.

The logic that toposes model is typically intuitionistic. What properties must a topos have to model classical logic? Boolean toposes.

theories have classifying toposes

For us

Using ccc's we can model propositional logic (intuitionistic), but we wish to add quantifiers.

Consider $P = \{x \mid x \text{ is a person}\}$.

Suppose we wanted to codify

"All tall people wear hats"

I need two predicates

$T(x) = x$ is tall

$H(x) = x$ wears a hat.

$$\forall x \, (T(x) \Rightarrow H(x))$$

x ranges over P. $\forall x \in P$.

$T(x)$ and $H(x)$ correspond to subsets of P.

Consider the predicate "$<$" less than

$x < y$ is modelled as a subset of $\mathbb{R} \times \mathbb{R}$

$$\forall x \exists y \; x < y.$$

$x + y = z$ would correspond to $\subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

So to think about predicates, we need to think about subsets, or for us, subobjects.

**First guess**

Def'n: An arrow $f: X \to Y$ is monic if

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

$$\forall g,h \quad g;f = h;f \Rightarrow g = h$$

So our first guess would be that a subobject is a monic arrow, since in Sets the map

$$U \subseteq X$$

corresponding to an inclusion is monic.

But this is wrong. Why? If $X$ is a finite set with $|X| = n$, there are $2^n$ subsets. But there are a proper class of ~~object~~ monic arrows with codomain $X$.

$$\begin{tikzcd} Z_1 = \{*_1\} \arrow[r, mapsto, "g_1"] \arrow[d, "\cong"'] & \{x, y, z\} = X \\ Z_2 = \{*_2\} \arrow[ur, mapsto, "g_2"'] & \end{tikzcd}$$

We have $Z_1 \cong Z_2$ and the iso commutes

Let $Mon(X)$ be the class of all monic arrows with $X$ as codomain. If

$$\begin{tikzcd} Y \arrow[dr, "f"] & \\ & X \\ Z \arrow[ur, "g"'] & \end{tikzcd} \qquad f, g \in Mon(X)$$

say $f \cong g$ if $\exists$ an isomorphism $i : Y \to Z$

s.t.

$$\begin{tikzcd} Y \arrow[dr, "f"] \arrow[dd, "i"'] & \\ & X \\ Z \arrow[ur, "g"'] & \end{tikzcd}$$

this diagram commutes.

Lemma: This is an equivalence relation on $Mon(X)$.

Def'n: Call the equivalence classes of this ER the <u>subobjects of</u> $X$. Denote it $Sub(X)$.

Remark. In general, it could still be a proper class. But for toposes, it will be a set.

Lem In Sets, $Sub(X) \cong \mathcal{P}(X)$

We'll need the following lemma, which holds in any category.

Lemma: The pullback of a monic is a monic, i.e. if the following is a pullback diagram

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

and $f$ is monic, so is $f'$

Proof: If you're never done this, do it. $\square$

Let $2 = \{0,1\}$.

Lemma: For all sets $X$, there is an isomorphism

$$Sub(X) \cong Hom_{Set}(X, 2)$$

If $B \subseteq X$, define

$$\chi_B : X \to 2 \text{ by}$$

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if not} \end{cases}$$

Lemma: In Sets, the following diagram is a pullback, for $B \subseteq X$

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "!"'] & X \arrow[d, "\chi_B"] \\ 1 \arrow[r] & 2 \end{tikzcd}$$

$* \mapsto 1 \in 2$

$2$ is called a <u>subobject classifier</u> or <u>object of truth-values</u>

$$2 = \{F, T\}$$

Any topos has one of these, denoted $\Omega$, but it can be much more complicated than $2$. Typically it's not a boolean algebra, although it can be.

It's always a <u>Heyting algebra</u>.

Defn: A Heyting algebra is a bounded lattice, i.e. a poset with $0, 1, \wedge, \vee$, and an operator

$$a \to b$$

which satisfies

$$a \wedge b \leq c \text{ iff } b \leq a \to c$$

(So it's a ccc as a category.)

Lemma: If $B$ is a boolean algebra, $B$ is a Heyting algebra with

$$a \to b = \neg a \vee b$$

Nonboolean example: If $X$ is a topological space, $\mathcal{O}(X)$ = set of opens in $X$, ordered under inclusion.

If $U, V \in \mathcal{O}(X)$

$$U \Rightarrow V = (U^c \cup V)^\circ$$

complement of $U$

The algebra of truth values of any model of intuitionistic logic is a Heyting algebra.

Heyting algebras satisfies

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

In fact $a \wedge (\ )$ will also satisfy

$$a \wedge \bigvee_i b = \bigvee_i (a \wedge b_i) \quad \text{for any}$$

infinitary suprema that exist.

Why? $a \wedge (\ )$ has a right adjoint & left adjoints preserve ~~suprema~~ colimits, i.e. suprema.

Any Heyting algebra has a <u>pseudo-complement</u> $\neg a$, defined as $\neg a = a \Rightarrow 0$.

We always have $a \leq \neg\neg a$, but not

$$a = \neg\neg a \quad \text{in general}$$

We always have

$$a \wedge \neg a = 0$$

but not $a \vee \neg a = 1$ in general.

Defn: A <u>complete Heyting algebra</u> or

cHa.. is a heyting algebra

with all sups. Alternatively, a cHa can be defined as a complete sup lattice such that

$$x \wedge \bigvee_i b_i = \bigvee_i x \wedge b_i$$

Then one can define

$$a \to b = \bigvee \{ c \mid a \wedge c \leq b \} .$$

The standard example of a cHa is $\mathcal{O}(X)$ for $X$ a space. The <u>pseudo complement</u> of $U$ is the interior of the complement.

<u>EX</u>: Find a space $X$ and an open $U$ s.t. $\neg\neg U \neq U$. We always have

$$U \subseteq \neg\neg U \qquad \square$$

There ~~is~~ is an adjunction between

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left] & \mathrm{cHa} \arrow[l, bend left] \end{tikzcd}$$

See Johnstone - Stone Spaces

Def'n: 6.4 of Kostecki

A subobject classifier in $\mathcal{E}$ is an object $\Omega$ in $\mathcal{E}$ together with an arrow

$$\top : 1 \to \Omega$$

s.t. $\forall$ monic arrow $m : B \to A$, $\exists$ a unique arrow $\chi_m : A \to \Omega$ s.t.

$$\begin{tikzcd} B \arrow[r, hook, "m"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\chi_m"] \\ 1 \arrow[r, "\top"'] & \Omega \end{tikzcd}$$

is a pullback.

Lemma 6.5 $\chi_m$ is independent of choice of representative of equivalence class. In other words if $m \simeq m'$, then

$$\chi_m = \chi_{m'}$$

Def'n 6.8 Let $\mathcal{C}$ be a ccc with subobject classifier. The object

$$P(A) = \Omega^A$$

is called the <u>power object</u> of A

<u>Def'n</u>: A category is an <u>elementary topos</u> if it is:

- finitely complete
- cartesian closed
- subobject classifier.

<u>Lemmas</u>

1) The above implies $\mathcal{C}$ is finitely cocomplete

2) If $\mathcal{C}$ is finitely complete with subobject classifier and objects of the form $\Omega^A$ $\forall A$, then $\mathcal{C}$ is cartesian closed

<u>Examples of toposes</u>

1) Sets

2) FinSets

3) G-Sets, G a group

$\Omega$ is the 2 point set with trivial G-action.

4) If $\mathcal{C}$ is any small category

$\hat{\mathcal{C}} = Set^{\mathcal{C}^{op}}$ the category of <u>presheaves</u> on $\mathcal{C}$ is a topos.

$$\Omega : \mathcal{C}^{op} \to Set$$

$$\Omega(a) = Sub(Hom_{\mathcal{C}}(-, a))$$

~~set~~ ~~subg~~ subfunctor of the representable functor.

5) If X is a topological space, the category of Sheaves on X is a topos.

# Some History

Grothendieck - doing algebraic geometry, realized he needed to generalize the notion of space. So

$$(X, \mathcal{O}(X)) \text{ became } (\underset{\substack{\text{small}\\\text{category}}}{\mathcal{C}}, \underset{\substack{\text{"site"}\\\text{on } \mathcal{C}}}{S(\mathcal{C})})$$

The notion of site allows one to define sheaf. Lots of important results followed from this idea. A <u>(Grothendieck) topos</u> is a category of sheaves on some site. This is not a satisfactory definition, since different sites can induce the same topos. An intrinsic definition was given by Giraud. It's complicated.

Every Grothendieck topos is an elementary topos. Converse is false. The category of finite sets is not Grothendieck, since Grothendieck toposes are cocomplete.

## References

Barr-Wells: Toposes, Triples & Theories
MacLane-Moerdijk: Sheaves In Geometry & Logic
Articles by Leinster & Kostecki