Lawvere Theories of Definable Operations

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The Isotropy Group

- Let $\mathcal{C}$ be a category. We have an isotropy group functor $\mathfrak{Z} : \mathcal{C} \to \mathbf{Grp}$, defined by Prof. Hofstra and his colleagues.
- $\mathfrak{Z}C$ captures the definable automorphisms of $C \in \mathcal{C}$.
- In his thesis, Jason Parker obtained a syntactical characterization of the isotropy group when $\mathcal{C}$ is the category of models of an algebraic theory (actually, a quasi-equational theory).

Generalizing the Isotropy Group

- We generalize to an isotropy theory functor $\mathfrak{Z} : \mathcal{C} \to \mathbf{Law}$, where $\mathcal{C}$ is required to have finite products.
- Goal: Describe the definable homomorphisms between powers of $C \in \mathcal{C}$.
- The isotropy theory captures aspects of algebraic theories which are invisible to the isotropy group. For example, the isotropy group for commutative rings is trivial, but not so for the isotropy theory.

Plan

1. Lawvere theories
2. The isotropy theory
3. Algebraic theories
4. The isotropy of algebraic theories
5. Applications
   - Groups
   - Abelian groups
   - Modules
   - Commutative rings

Lawvere Theories

Arose from Lawvere's search for a more invariant description of algebraic theories.

Definition

A **Lawvere theory** $\mathcal{C}$ is a category with finite products, together with a choice of an object $C \in \mathcal{C}$ and specified product diagrams $c_i : C^n \to C$ for each $n \in \mathbb{N}$, such that each object of $\mathcal{C}$ is isomorphic to a power of C.

Lawvere theories

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be Lawvere theories with chosen objects C and D, respectively. A **morphism of Lawvere theories** is a functor $F : \mathcal{C} \to \mathcal{D}$ which strictly preserves the specified products:

$$(FC^n \overset{Fc_i}{\to} FC) = (D^n \overset{d_i}{\to} D).$$

Lawvere theories and their morphisms form a category **Law**.

The Lawvere Theory Associated to an Object

Let $\mathcal{C}$ be a category with finite products and assume we have specified product diagrams $c_i : C^n \to C$.
The Lawvere theory LC has objects the natural numbers, with hom-sets

$$LC(n, m) = \mathcal{C}(C^n, C^m).$$

Thus LC is equivalent to the full subcategory of $\mathcal{C}$ on the objects C^n, $n \in \mathbb{N}$.
Is the assignment $C \mapsto LC$ functorial? Not in general, but...

The Lawvere Theory Associated to an Object

Can be made functorial on the maximal subgroupoid $\mathcal{G}$ of $\mathcal{C}$.
Define $L : \mathcal{G} \to \mathbf{Law}$.
Given $f : C \to D$, we have a functor $Lf : LC \to LD$ with $(Lf)n = n$, and $(Lf)g$ is the unique arrow making the following diagram commute:

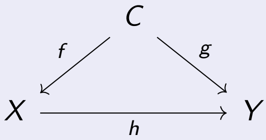
$$\begin{array}{ccc} C^n & \xrightarrow{g} & C^m \\ {\scriptstyle f^n}\downarrow\wr & & {\scriptstyle f^m}\downarrow\wr \\ D^n & \xrightarrow[(Lf)g]{} & D^m \end{array}$$

Soon, we'll obtain a functor $\mathcal{C} \to \mathbf{Law}$. But before that...

Reminder: Coslice Categories

Definition

The **coslice category** $(C \downarrow \mathcal{C})$.
Objects: arrows $f : C \to X$ from C in $\mathcal{C}$
Arrows:



There is a canonical projection functor $P_C : (C \downarrow \mathcal{C}) \to \mathcal{C}$.

The Isotropy Theory Functor

The functor $P_C^n = (-)^n \circ P_C : (C \downarrow \mathcal{C}) \to \mathcal{C}$ is the n-th power of P_C in $\mathcal{C}^{(C\downarrow\mathcal{C})}$. (Products in this category are defined pointwise.)

Definition

The **isotropy theory functor** $\mathfrak{Z} : \mathcal{C} \to \mathbf{Law}$ is defined by $\mathfrak{Z}C = LP_C$. Thus $\mathfrak{Z}C(n, m) = \mathrm{Nat}(P_C^n, P_C^m)$.
Given $h : C \to D$, we define a morphism of Lawvere theories $\mathfrak{Z}h : \mathfrak{Z}C \to \mathfrak{Z}D$. On objects, $(\mathfrak{Z}h)n = n$. To define $\mathfrak{Z}h$ on arrows, let $\tau : P_C^n \Rightarrow P_C^m$ and $(D \xrightarrow{f} X) \in (D \downarrow \mathcal{C})$. We let

$$(Zh)\tau : P_D^n \Rightarrow P_D^m, \qquad ((Zh)\tau)_f = \tau_{f\circ h}.$$

$\mathfrak{Z}C$ is determined by all hom-sets of the form $\mathfrak{Z}C(n, 1)$.

The Isotropy Theory Functor

More explicitly, an arrow of $\mathfrak{Z}C$, say $\tau : P_C^n \Rightarrow P_C^m$, is just a family of arrows $\tau_f : X^n \to X^m$ for each $f : C \to X$ in $\mathcal{C}$, such that every commutative triangle

$$\begin{array}{ccc} & C & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \xrightarrow[k]{} & Y \end{array}$$

induces a commutative square

$$\begin{array}{ccc} X^n & \xrightarrow{\tau_f} & X^m \\ {\scriptstyle k^n}\big\downarrow & & \big\downarrow{\scriptstyle k^m} \\ Y^n & \xrightarrow[\tau_g]{} & Y^m \end{array}$$

The Isotropy Theory Functor

There is a canonical **comparison morphism** $\nu_C : \mathfrak{Z}C \to LC$;
$\nu_C n = n$;
$\tau : P^n_C \Rightarrow P^m_C$ implies $\nu_C \tau = \tau_{1_C}$.
Arrows in the image of ν_C are called isotropy arrows. We think of them as definable morphisms.

Algebraic Theories

Definition

A **signature** Σ is a set of operation symbols, each with their own specified arity $n \in \mathbb{N}$.
A Σ-**structure** is a set A equipped with an operation $f^A : A^n \to A$ for each operation symbol f of arity n in Σ.

A Σ-term t in the variables $y_1, \ldots, y_n$ defines a function $t^A : A^n \to A$ by $(y_1, \ldots, y_n) \mapsto t$.

Definition

An **algebraic theory** $T = (\Sigma, E)$ consists in a signature Σ and a set E of equations between Σ-terms.
A Σ-structure A is a **model** of T iff $s^A = t^A$ for each equation $(s = t) \in E$.

Algebraic Theories

Definition

A **homomorphism** of T-models $h : A \to B$ is a function which commutes with all operations of T:

$$h \circ f^A = f^B \circ h^n.$$

There is a category T-**Mod** of T-models and T-homomorphisms. Algebraic theories always have a free-forgetful adjunction:

$$F \dashv P$$

We denote the free T-model on a set X by $\langle X \rangle$. It's the quotient of the set of Σ-terms by the smallest equivalence relation realising all the equations of T.

The Isotropy of Algebraic Theories

We give a syntactic characterization of $\mathfrak{Z} : T\text{-}\mathbf{Mod} \to \mathbf{Law}$.

- Adjoining indeterminates $x_1, \ldots, x_n$ to a model $A \in T\text{-}\mathbf{Mod}$: $A\langle \vec{x} \rangle = A \amalg \langle \vec{x} \rangle$.
- Elements of $A\langle \vec{x} \rangle$ are (equivalence classes of) terms with parameters in A.
- A term $t \in A\langle x_1, \ldots, x_n \rangle$ induces a function $t^A : A^n \to A$ by $(x_1, \ldots, x_n) \mapsto t$.
- If $t \in A\langle x_1, \ldots, x_n \rangle$ and $h : A \to B$ is a homomorphism then t_h is obtained from t by applying h to all elements of A in the term t.
- If $(\vec{t}\,) \in (A\langle \vec{x} \rangle)^m$ then $(\vec{t}\,)_h = ((t_1)_h, \ldots, (t_m)_h)$.

The Isotropy of Algebraic Theories

Consider the Lawvere theory $\mathcal{T}_A$ whose objects are the natural numbers and whose arrows $n \to m$ are m-tuples of terms in $A\langle x_1, \ldots, x_n \rangle$.

- Composition: $$n \xrightarrow{(t_1,\ldots,t_m)} m \xrightarrow{(s_1,\ldots,s_l)} l, \qquad n \xrightarrow{(s_1[t_j/x_j],\ldots,s_l[t_j/x_j])} l$$
- Identities: $1_n = (x_1, \ldots, x_n)$
- Products: $x_i : n \to 1$

This is the Lawvere theory of T-models with parameters in A.

The Isotropy of Algebraic Theories

Definition

The functor $\mathfrak{G} : T$-**Mod** $\to$ **Law** is as follows.
$\mathfrak{G}A$ is the subcategory of $\mathcal{T}_A$ with the same objects and with arrows only those terms $(t_1, \ldots, t_m) : n \to m$ such that $(\vec{t}\,)^B_h$ is a T-homomorphism for all homomorphisms $h : A \to B$. Intuitively, $(\vec{t}\,)$ defines a homomorphism $B^n \to B^m$ for any model B with parameters in A.
If $f : A \to C$ then $\mathfrak{G}f : \mathfrak{G}A \to \mathfrak{G}C$ with

$$(\mathfrak{G}f)n = n, \qquad (\mathfrak{G}f)(\vec{t}\,) = (\vec{t}\,)_f.$$

The Isotropy of Algebraic Theories

Theorem

There is a natural isomorphism $\mathfrak{Z} \cong \mathfrak{G}$.

Most importantly, $\mathfrak{G}A(n,1) \cong \mathfrak{Z}A(n,1)$ via $t \mapsto \tau$ with $\tau_h = t_h^B$.
Furthermore, $\nu_A \tau = t_{1_A}^A = t^A : A^n \to A$.

The Isotropy of Algebraic Theories

Definition

Let f be an operation symbol of arity l, and let $t \in A\langle \vec{x} \rangle$. Also let $(\vec{y}) = (y_{11}, \ldots, y_{ln})$ be a list of $l \cdot n$ variables. Then t **commutes generically** with f iff

$$f^{A\langle \vec{y} \rangle}(t[y_{1i}/x_i], \ldots, t[y_{li}/x_i]) = t[f^{A\langle \vec{y} \rangle}(y_{1i}, \ldots, y_{li})/x_i].$$

Think of $t \in A\langle x_1, \ldots, x_n \rangle$ as the function $t_h^B : B^n \to B$, which we want to commute with all operations f, so that it's a homomorphism.

The Isotropy of Algebraic Theories

As examples, consider the group operations:

- $t[y_{1i}/x_i]t[y_{2i}/x_i] = t[y_{1i}y_{2i}/x_i]$
- $(t[y_{1i}/x_i])^{-1} = t[y_{1i}^{-1}/x_i]$
- $e = t[e/x_i]$

Theorem

Let $t \in A\langle \vec{x} \rangle$. Then $t \in \mathfrak{G}A(n, 1)$ iff t commutes generically with all operation symbols of T.

Thus $\mathfrak{Z} \cong \mathfrak{G}$ with $\mathfrak{G}A(n, 1)$ consisting of the terms $t \in A\langle x_1, \ldots, x_n \rangle$ which generically commute with all operations of T.

Application: Groups

Let T be the theory of groups, let $A \in \mathbf{Grp}$.
If $t \in \mathfrak{G}A(n, 1)$ then $t = e$ or $t = ax_i a^{-1}$.
This is very similar to the isotropy group, which has $t = axa^{-1}$.

Application: Abelian Groups

Let T be the theory of abelian groups, let $A \in \mathbf{Ab}$.

$$\mathfrak{G}A(n,1) = \{\, k_1 x_1 + \cdots + k_n x_n \mid k_1, \ldots, k_n \in \mathbb{Z} \,\},$$

which contains more than the isotropy group $\{\, x, -x \,\}$.

Application: Modules

Let R be a ring, let T be the theory of R-modules, let A be an R-module.

$$\mathfrak{G}A(n,1) = \{\, r_1 x_1 + \cdots + r_n x_n \mid r_1, \ldots, r_n \in ZR \,\},$$

where $ZR = \{\, s \in R \mid rs = sr \text{ for all } r \in R \,\}$ is the center of R. The correspondence

$$\mathfrak{G}A(n,m) \leftrightarrow (ZR)^{m \times n}$$
$$\Big(\sum_i r_{1i} x_i, \ldots, \sum_i r_{mi} x_i\Big) \leftrightarrow (r_{ji})$$

induces an isomorphism of categories $\mathfrak{G}A \cong \mathbf{Mat}(ZR)$.

Application: Commutative Rings

Let T be the theory of commutative rings, let $A \in \mathbf{CRing}$. Elements of $\mathfrak{G}A(n, 1)$ have the form $\sum_{i=1}^{n} b_i x_i$ where $b_1, \ldots, b_n$ is a system of orthogonal idempotents summing to 1:

$$b_i b_j = \begin{cases} b_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad \sum_{i=1}^{n} b_i = 1.$$

These correspond to decompositions of A as a direct sum of left ideals (equivalently, A-submodules).