Skew Braces & Hopf algebras in the category SupLat

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13 Nov 2021
Category Theory Novemberfest

Based on: arXiv:2001.08673 [G1], arXiv:2005.07183 [G2]

background

$$\begin{tikzcd}
\text{Linear YBE Solutions} \arrow[r, bend right, "\text{FRT construction}"'] & \text{(Co)-quasitriangular bialgebras} \arrow[l, bend right, "\text{Rep theory}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Set-theoretical YBE Solutions} \arrow[r, bend right, "\text{Universal Group}"'] & \textbf{\textit{Skew Braces}} \arrow[l, bend right, "\text{Bachiller's result}"']
\end{tikzcd}$$

Question: Can skew braces be viewed as Hopf algebras in a reasonable category?

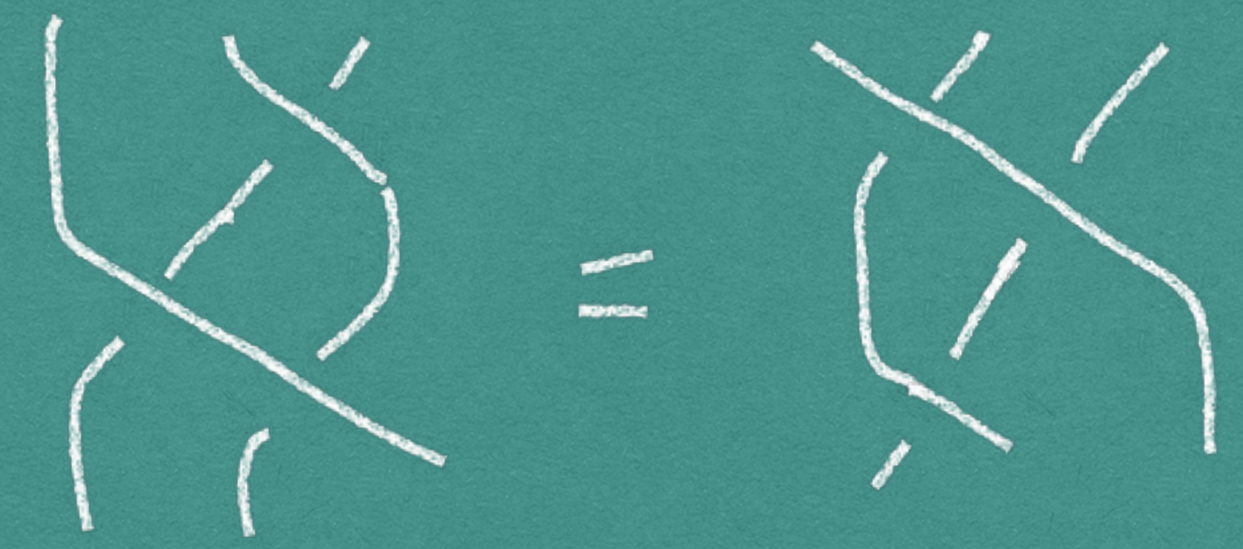
YBE background

Let $(\mathscr{C}, \otimes, 1)$ will denote a monoidal category.

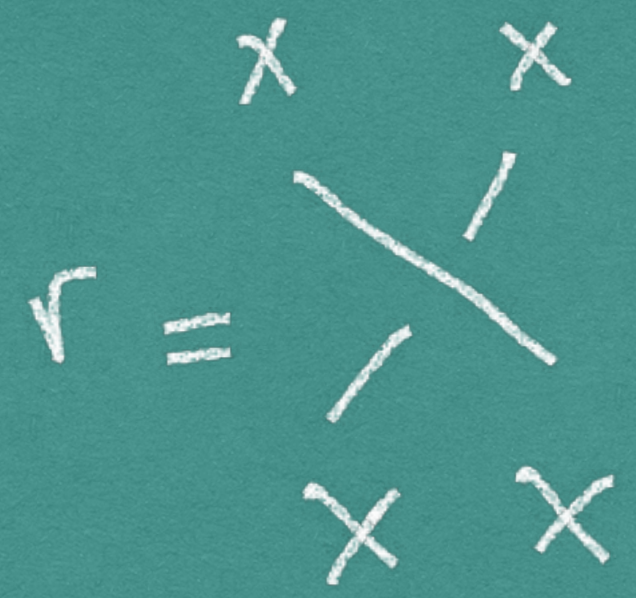
We can define a ***braided object*** in any such $\mathscr{C}$ as a pair $(X, r : X \otimes X \to X \otimes X)$ satisfying

$$(\mathrm{id}_X \otimes r)(r \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes r) = (r \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes r)(r \otimes \mathrm{id}_X)$$

Yang-Baxter Equation



diagrams up to down



objects → rows

morphisms → "colums" ↓

Dualizable objects $X, X^\vee$

$X \otimes X^\vee \to 1$

$1 \to X^\vee \otimes X$

YBE background (Upgraded)

a ***braided object*** in $(\mathscr{C}, \otimes, 1)$ $\overset{1:1}{\longleftrightarrow}$ A strict monoidal functor $\mathscr{B} \to \mathscr{C}$

$\mathfrak{B}$ "Smallest braided category", tensor generated by one object and a morphism satisfying YBE

How do we construct a Hopfy object from a YBE solution?

Tannaka-Krein Reconstruction:

$$\begin{tikzcd} \omega : \mathcal{D} \arrow[r, "\otimes"] \arrow[dr] & \mathcal{C} \\ & \mathcal{C}^{rig} \arrow[u] \end{tikzcd}$$

$$H_\omega = \int^{d \in \mathcal{D}} \omega(d) \otimes \omega(d)^\vee$$

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr] & & \mathcal{C} \\ & {}^{H}\mathcal{C} \arrow[ur, "forg"'] & \end{tikzcd}$$

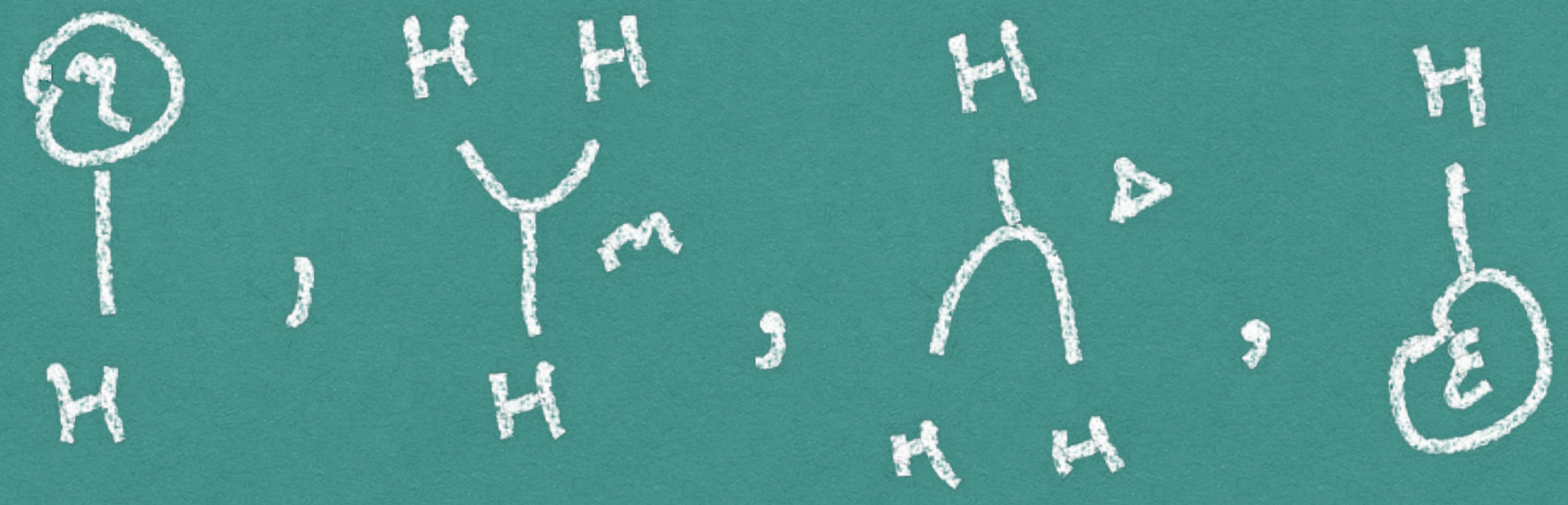
If $\mathscr{C}$ is braided (or even better, symmetric)

then it's a Hopf monoid/algebra

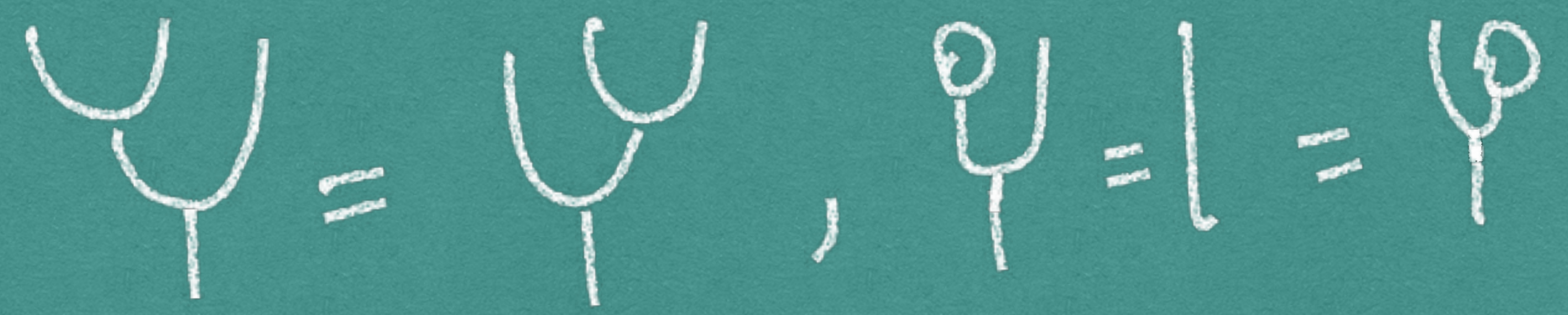
Hopf algebras in 1 min

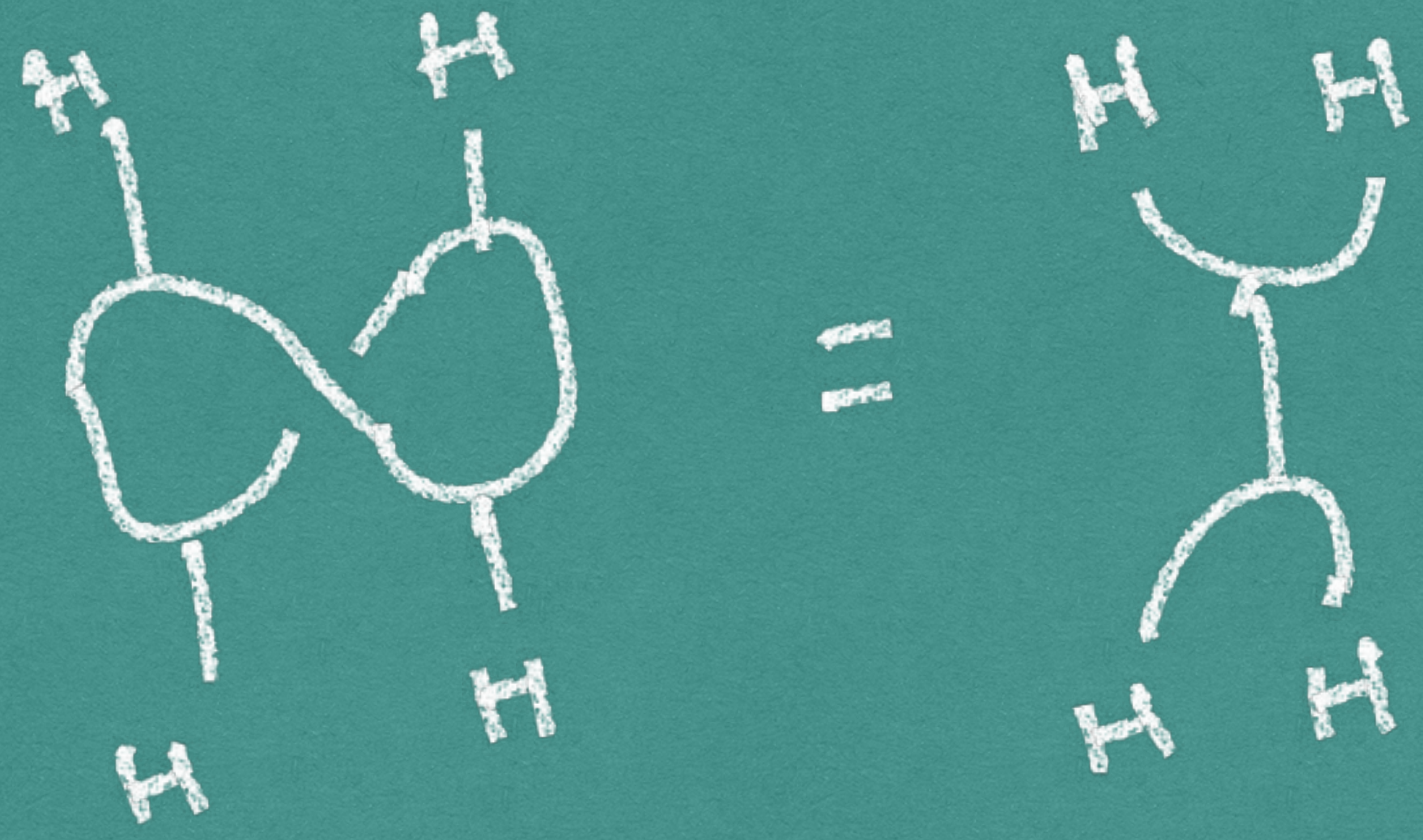
Need a braided/symmetric category $(\mathscr{C}, \otimes, 1)$

A monoid (H, m, η) + comonoid (H, Δ, ϵ)



+ bialgebra condition says its a (comonoid) monoid in the *monoidal* category of (monoid) comonoids in $\mathscr{C}$





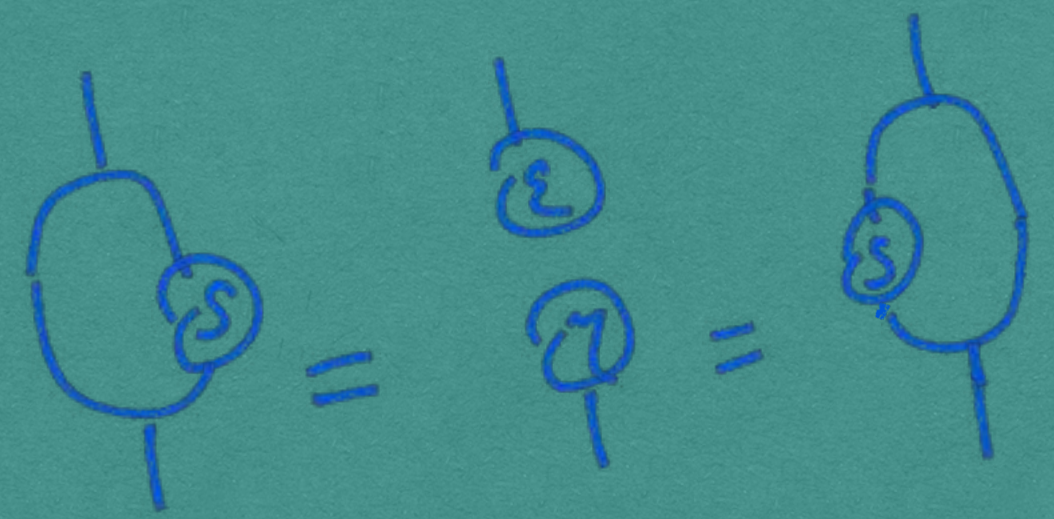
Category of comodules, modules are monoidal +

${}^{H}\mathcal{C} \,/\, {}_{H}\mathcal{C} \xrightarrow{\text{forg}} \mathcal{C}$ are monoidal

Hopf algebras in 1 min

If H is a Hopf monoid

$S: H \longrightarrow H$



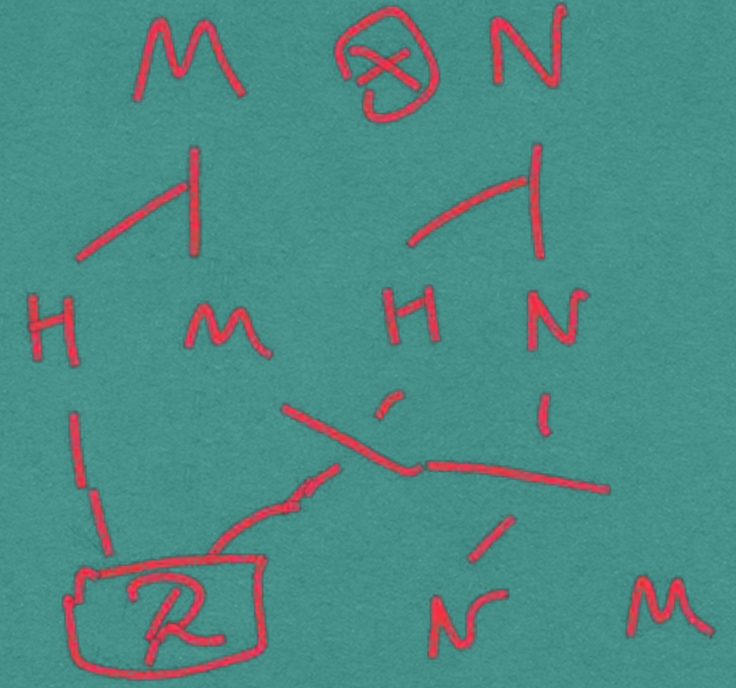
${}^{H}\mathcal{C} / {}_{H}\mathcal{C} \xrightarrow{\text{forg}} \mathcal{C}$

Lifts the closed structure of $\mathscr{C}$

(If one exists)

I'm always assuming antipode is invertible!

A co-quasitriangular structure on H makes ${}^{H}\mathscr{C}$ braided.



$R: H \otimes H \longrightarrow 1$

$R \in H \otimes H$

R-matrix

Dualise

Hopf algebras in 1 min

$${}^{H}\mathcal{C}/_{H}\mathcal{C} \xrightarrow{\text{forg}} \mathcal{C}$$

If H is a Hopf monoid (has extra condition)

Lifts the closed structure of $\mathscr{C}$

A co-quasitriangular structure on H makes ${}^{H}\mathscr{C}$ braided.

(If one exists)

$$\begin{tikzcd} \omega : \mathcal{D} \arrow[rr, "\otimes"] \arrow[dr] & & \mathcal{C} \\ & \mathcal{C}^{rig} \arrow[ur] & \end{tikzcd}$$

$$H_\omega = \int^{d \in \mathcal{D}} \omega(d) \otimes \omega(d)^\vee$$

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr, hook] & & \mathcal{C} \\ & {}^{H}\mathcal{C} \arrow[ur, "\text{forg}"'] & \end{tikzcd}$$

For reconstruction purposes, if $\mathscr{D}$ is rigid, then H_ω is a Hopf algebra

if $\mathscr{D}$ is *braided*, then H_ω obtains a co-quasitriangular structure

background

Linear YBE Solutions on f.d. Vector space $\underset{\text{Rep theory}}{\overset{\text{FRT construction}}{\rightleftarrows}}$ (Co)-quasitriangular bialgebras

$\omega : \mathcal{B} \longrightarrow \mathrm{Vec} \Rightarrow H_\omega = A(R)$ FRT bialgebra

⇝ (Fadeev-Reshitikhin-Takhtajan)

Rigid Upgrade

Assume $(V, R : V \otimes V \to V \otimes V)$ is *bi-invertible*

$\omega : \mathcal{B}_{rig} \longrightarrow \mathrm{Vec}$ ⇝ $GL(R)$ Lyubashenko

Path to enlightenment

Set-theoretical YBE solution

$\omega : \mathcal{B} \xrightarrow{\otimes} \mathrm{Set} \hookrightarrow \mathrm{Rel} \hookrightarrow \mathrm{SupLat}$

Simplest idea: apply FRT to (Set, $\times$,1)

Only dualizable object is 1 + CQ problem

$X \overset{f}{\not\to} Y \qquad f \subseteq X \times Y$

Naive Solution: embed into (Rel, $\times$,1)

Not enough colimits for the coend!

$$\begin{cases} \mathrm{coev}: 1 \not\to X \times X \\ \{(1,x,x) \mid x \in X\} \subseteq 1 \times X \times X \\ \mathrm{ev}: X \times X \not\to 1 \\ \{(x,x,1) \mid x \in X\} \subseteq 1 \times X \times X \end{cases}$$

Correct Solution: SupLat

SupLat

Reference: Joyal-Tierney: An extension of the Galois theory of Grothendieck

- Objects: partially ordered sets $(\mathscr{L}, \leq)$, where any subset $S \subseteq \mathscr{L}$, has a least upper bound, $\bigvee S$, called ***joins***
- Morphisms: join-preserving maps

Notation: $\vee_{i\in I} a_i$ for $\vee \{a_i \mid i \in I\}$

- All objects in SupLat have ***meets***: (they're complete lattices!)

$$\wedge S = \vee \{a \mid a \leq s \ , \forall s \in S\}$$

- Free Lattices: For any set X, its power-set $\mathscr{P}(X)$, with $\vee = \cup$ is a complete lattice.
- SupLat is complete and co-complete

$\mathscr{M} \otimes \mathscr{N}$ = Quotient of $\mathscr{P}(\mathscr{M} \times \mathscr{N})$ by relations

- SupLat is symmetric monoidal closed:

$$\{(\vee_{i\in I} m_i, n)\} = \cup_{i\in I} \{(m_i, n)\}$$

$$\{(m, \vee_{i\in I} n_i)\} = \cup_{i\in I} \{(m, n_i)\}$$

SupLat

$$P(-):\ \begin{tikzcd} & \underline{\mathrm{Rel}} \arrow[dr] & \\ \underline{\mathrm{Sets}} \arrow[ur] \arrow[rr, "\otimes"] & & \underline{\mathrm{SupLat}} \end{tikzcd}$$

We have a faithful monoidal functor

Lemma. [G1] Dualisable objects in SupLat are free lattices.

$$\underbrace{\left\{\begin{array}{c}\text{invertible}\\ \text{set-theoretical}\\ \text{YBE solutions}\end{array}\right\}}_{\text{Set}} \hookrightarrow \underbrace{\left\{\begin{array}{c}\text{invertible}\\ \text{YBE solutions}\\ \text{on dualizable objects}\end{array}\right\}}_{\text{SupLat}}$$

$\varepsilon : \mathcal{H} \to \{0,1\}$ $\varepsilon^{-1}(0) \rightrightarrows H \to Q$

Thm $Q \cong \mathcal{P}(\text{Group})$

Main Theorems in [G1]

From any Hopf algebra H in SupLat, we can construct a group called its *remnant* R(H)

Any co-quasitriangular structure on H gives a skew brace structure on R(H)

Any skew brace can be recovered in this way! (non-uniquely)

The remnant of the SupLat-FRT of (($\mathscr{P}(X)$, $\mathscr{P}(r)$) of a solution (X, r) is a the universal group

(X, r) Set $\rightsquigarrow$ $(\mathcal{P}(X), \mathcal{P}(r))$ SupLat $\overset{\text{FRT}}{\rightsquigarrow}$ $H_\omega \rightsquigarrow R(\mathcal{H}_\omega)$ Universal Group

Applications:

1) We can ***rewrite*** the theory of skew braces in terms of Hopf algebras!

[G1] Transmutation: given a CQHA H,

We obtain a new product on H $\Rightarrow$ Modern definition of skew braces

$$(G, ., e, ^{-1}) + r \qquad (G, ., \star\ )$$

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$$\begin{tikzcd} \text{Set-theoretical YBE Solutions} \arrow[r, bend right, "\text{Universal Group}"'] & \text{Skew braces} \arrow[l, bend right, "\text{Bachiller's result}"'] \end{tikzcd}$$

Q Can we re-interpret Bachiller's result?

Applications:

2) We can ***apply*** Hopf algebra techniques in the theory of skew braces

[G2] I define the notion of Drinfeld twists for skew braces!

Future work: Apply Co-double bosonisation to get new skew braces!

3) Combinatorial knot Invariants = Quantum invariants?

skew braces ⊆ biquandles

$U_q(g)$, (co)-quasitriangular Hopf algebras

Questions for Category Theorists

Is the remnant construction part of an adjunction?

Why SupLat?

Good monoidal functors between SupLat and Vec?

$$\begin{tikzcd}
(X,r)\in\mathrm{Set} \arrow[rr, "\text{linearise}"] \arrow[d] & & (\mathbb{K}.X,\, r)\in\mathrm{Vec} \arrow[d, "\mathrm{FRT}"] \\
\mathrm{SupLat} \arrow[r, "\mathrm{FRT}"] & H_r & \text{linear Hopf}
\end{tikzcd}$$

Thank you
for your Attention!

:)

References:

[G1] A Ghobadi, Skew Braces as Remnants of Co-quasitriangular Hopf Algebras in SupLat, Journal of Algebra 2021, arXiv:2009.12815
[G2] A Ghobadi, Drinfeld Twists on Skew Brace, arXiv:2105.03286