The Equivariant Fundamental Double Groupoid

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- A **double category** is an internal category in **Cat**,

$$\mathbf{C}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathbf{C}_0 \ .$$

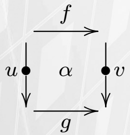
- It has
  - objects (objects of $\mathbf{C}_0$),
  - vertical arrows (arrows of $\mathbf{C}_0$), denoted $d_0(v) \overset{v}{\longrightarrow\!\!\bullet\!\!\rightarrow} d_1(v)$,
  - horizontal arrows (objects of $\mathbf{C}_1$), denoted $s(f) \xrightarrow{f} t(f)$,
  - double cells (arrows of $\mathbf{C}_1$), denoted

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow v \\ A' & \xrightarrow[f']{} & B' \end{array}$$

where $d_0(\alpha) = f$, $d_1(\alpha) = f'$, $s(\alpha) = u$, and $t(\alpha) = v$.

Examples

1. For any 2-category $\mathcal{C}$, $\mathbb{Q}(\mathcal{C})$ is the double category of quintets in $\mathcal{C}$, with double cells



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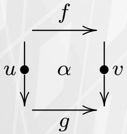

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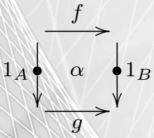

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for each $\alpha\colon vf \Rightarrow gu$ in $\mathcal{C}$.

2. For any 2-category $\mathcal{C}$, $\mathbb{H}(\mathcal{C})$ is the double category with double cells

$$\begin{array}{ccc} & \xrightarrow{f} & \\ 1_A \big\downarrow & \alpha & \big\downarrow 1_B \\ & \xrightarrow[g]{} & \end{array}$$

for each $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$.

3. The double category $\mathbb{V}(\mathcal{C})$ is defined analogously.

The category **DblCat**

The category **DblCat** of double categories has:

- objects: double categories $\mathbb{C}, \mathbb{D}, \ldots$;
- arrows: double functors $F, G, \ldots$;
- 2-cells: these come in two flavours:

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  - vertical transformations $\gamma\colon F \mathrel{\bullet\!\!\Rightarrow} G\colon \mathbb{C} \rightrightarrows \mathbb{D}$ given by

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle\gamma_A}\,\bullet\!\!\downarrow & \gamma_h & \bullet\!\!\downarrow\,{\scriptstyle\gamma_B} \\ GA & \xrightarrow[Gh]{} & GB \end{array} \qquad \text{for each } h\colon A \to B \text{ in } \mathbb{C}$$

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    functorial in the horizontal direction and natural in the vertical direction.
  - horizontal transformations $\nu\colon F \Longrightarrow G$ are defined dually;
- modifications given by a family of double cells.

The category **DblCat** - Properties

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- **DblCat**$_v$ (resp. **DblCat**$_h$) is the 2-category with vertical (resp. horizontal) transformations.
- So lax limits have typically been taken in the 2-category **DblCat**$_v$ or **DblCat**$_h$ with laxity in one direction.

Diagrams in **DblCat**

To define a diagram of double categories indexed by a double category $\mathbb{D}$:

- Send objects of $\mathbb{D}$ to double categories;
- Send both horizontal and vertical arrows to double functors;
- Send double cells to *vertical* transformations.

So an indexing double functor is a double functor

$$\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v)$$

We will also refer to indexing double functors as **vertical double functors**

$$\mathbb{D} \overset{v}{\longrightarrow} \mathbf{DblCat}.$$

The Double Grothendieck Construction: Objects and Arrows

Let $\mathbb{D} \xrightarrow{F} \mathbf{DblCat}$ be a vertical double functor. The **double category of elements**, $\mathbb{G}\mathrm{r}\, F = \int_{\mathbb{D}} F$, is defined by:

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$$(C, x) \xrightarrow{(u,\rho)} (C', x'),$$

  where $C \xrightarrow{u} C'$ in $\mathbb{D}$ and $Fux \xrightarrow{\rho} x'$ in FC'.

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- Horizontal arrows:

$$(C, x) \xrightarrow{(f,\varphi)} (D, y),$$

  where $C \xrightarrow{f} D$ in $\mathbb{D}$, and $Ffx \xrightarrow{\varphi} y$ in FD.

The Double Grothendieck Construction: Double Cells

- Double cells: $$\begin{array}{ccc} (C,x) & \xrightarrow{(f,\varphi)} & (D,y) \\ {\scriptstyle (u,\rho)}\big\downarrow\bullet & (\alpha,\Phi) & \bullet\big\downarrow{\scriptstyle (v,\lambda)} \\ (C',x') & \xrightarrow[(f',\varphi')]{} & (D',y') \end{array}$$, where $\alpha\colon (u \;{}^{f}_{f'}\; v)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FD':

$$\begin{array}{ccc} FvFfx & \xrightarrow{Fv\varphi} & Fvy \\ {\scriptstyle (F\alpha)_x}\big\downarrow\bullet & & \big| \\ Ff'Fux & \Phi & \bullet\ \lambda \\ {\scriptstyle Ff'\rho}\big\downarrow\bullet & & \big\downarrow \\ Ff'x' & \xrightarrow[\varphi']{} & y' \end{array}$$

The Main Theorem

- There is a doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta \mathbb{G}\mathrm{r}\, F$ with the required universal property:

$$\lambda^* \colon \mathbf{DblCat}\left(\int_{\mathbb{D}} F, \mathbb{E}\right) \to \mathbb{L}\mathsf{C}\left(\int_{\mathbb{D}} F, \mathbb{E}\right)$$

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- Furthermore, $\int_{\mathbb{D}}$ extends to a functor of DblCat-categories

$$\mathrm{Hom}_v(\mathbb{D}, \mathbf{DblCat})_{d\ell} \to \mathbf{DblCat}/\mathbb{D}$$

  which is locally an isomorphism of double categories

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G) \cong (\mathbf{DblCat}/\mathbb{D})\left(\int_{\mathbb{D}} F \to \mathbb{D}, \int_{\mathbb{D}} G \to \mathbb{D}\right).$$

Work in progress

A versions of tom Dieck Fundamental **Double** Groupoid

For any group G the orbit category $\mathcal{O}_G$ is defined as follows:

- Objects: G/H where H is a closed subgroup of G;
- Arrows: G-equivariant maps

$$a\colon G/H \to G/K$$

where $H \subseteq aKa^{-1}$.

Note that arrows can be viewed as points in $(G/K)^H$; they can also be viewed as elements of G: conjugation by a after a canonical projection.

Let X be a G-space.

$$\mathcal{F} : \mathcal{O}_G \to \textbf{Cat}$$

- Objects: $\mathcal{F}(G/H) = \pi(X^H)$ the fundamental groupoid of X^H;
- Arrows:

$$\mathcal{F}a : \pi(X^K) \to \pi(X^H)$$

$$[\gamma] \longmapsto [a\gamma]$$

Orbit Category

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The tom Dieck orbit category $\mathcal{O}_G(X)$ is obtained as a quotient of a categorical Grothendieck construction $\int_{\mathcal{O}_G} \mathcal{F}$, we have:

- Objects: $(G/H, x)$ where $x \in X^H$;
- Arrows:

$$(G/H, x) \xrightarrow{(a,[\gamma])} (G/K, y)$$

where $[\gamma]$ is a homotopy class of path in X^H from x to ay.

If G is a Lie group, then $\mathcal{O}_G$ admits a 2-category structure:

- Objects: G/H where H is a closed subgroup of G;
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2-Category version of Orbit Category

If G is a Lie group, then $\mathcal{O}_G$ admits a 2-category structure:

- Objects: G/H where H is a closed subgroup of G;
- Arrows: G-equivariant maps

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where $H \subseteq aKa^{-1}$.

- 2-cells:

$$G/H \underset{b}{\overset{a}{\rightrightarrows}} G/K \qquad [s] \Downarrow$$

$[s]$ is a homotopy class of paths in $(G/K)^H$ from a to b.

2-Category version of Orbit Category

Let X be a G-space.

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- Arrows: $\mathcal{F}a : \pi(X^K) \to \pi(X^H)$
- 2-cells:

$$\mathcal{F}[s] : \mathcal{F}a \Rightarrow \mathcal{F}b$$

is a natural transformation and for $x_1 \xrightarrow{[\gamma]} x_2$ in X^K we have

$$\begin{array}{ccc} ax_1 & \xrightarrow{[a\gamma]} & ax_2 \\ {\scriptstyle [sx_1]}\downarrow & & \downarrow{\scriptstyle [sx_2]} \\ bx_1 & \xrightarrow[{[b\gamma]}]{} & bx_2 \end{array}$$

2-Category version of Orbit Category

The tom Dieck fundamental groupoid is obtained as a quotient of a categorical Grothendieck construction $\int_{\mathcal{O}_G} \mathcal{F}$.

- Objects: $(G/H, x)$ where $x \in X^H$;
- Arrows:

$$(G/H, x) \xrightarrow{(a,[\gamma])} (G/K, y)$$

  where $[\gamma]$ is a homotopy class of path in X^H from x to ay.
- 2-cells:

$$[\sigma] : (a, [\gamma]) \Rightarrow (b, [\eta])$$

  is a homotopy class of paths from a to b in $(G/K)^H$ s.t the following diagram commutes in X^H.

$$\begin{array}{ccc} x & \xrightarrow{[\gamma]} & ay \\ {\scriptstyle [id_x]}\Big| & & \Big\downarrow{\scriptstyle [\sigma y]} \\ x & \xrightarrow[{[\eta]}]{} & by \end{array}$$

Double Category version of Orbit Category

If G is a Lie group, then the orbit category admits a double category structure, denoted by $\mathbb{O}_G^2$:

- Objects: G/H where H is a closed subgroup of G;
- Horizontal arrows: G-equivariant maps

$$a\colon G/K_1 \to G/K_2$$

  where $K_1 \subseteq aK_2a^{-1}$.
- Vertical arrows:

$$G/K_1 \xrightarrow{[\theta]} G/L_1$$

  homotopy class of paths in G from e to $\theta(1)$ where $\theta(1)K_1\theta(1)^{-1} = L_1$.

- Double cells:

$$\begin{array}{ccc} G/K_1 & \xrightarrow{\ a\ } & G/K_2 \\ {\scriptstyle [\theta_1]}\bullet\big\downarrow & \xi & \big\downarrow\bullet{\scriptstyle [\theta_2]} \\ G/L_1 & \xrightarrow[\ b\ ]{} & G/L_2 \end{array}$$

ξ is a path from a to b such that

$$G/K_1 \xrightarrow{\theta_1(t)^{-1}\xi(t)\theta_2(t)} G/K_2$$

Double Category version of Orbit Category

Double category of fundamental groupoids, $\mathbb{Q}\pi(X)$

- Objects: $x \in X$;
- Horizontal and vertical arrows: homotopy class of paths in X;
- Double cells:

$$\begin{array}{ccc} x_1 & \xrightarrow{[u]} & x_2 \\ {\scriptstyle [\alpha_1]}\downarrow & H & \downarrow{\scriptstyle [\alpha_2]} \\ y_1 & \xrightarrow[{[v]}]{} & y_2 \end{array}$$

Double Category version of Orbit Category

$$\mathcal{F} : \mathbb{O}_G^1 \to DblCat$$

- Objects: $\mathcal{F}(G/K) = \mathbb{Q}\pi(X^K)$;
- Horizontal arrows:

$$\mathcal{F}a : \mathbb{Q}\pi(X^{K_2}) \to \mathbb{Q}\pi(X^{K_1})$$

  where $a : G/K_1 \to G/K_2$
- Vertical arrows:

$$\mathcal{F}\theta = \theta(1)^{-1} : \mathbb{Q}\pi(X^{L_1}) \to\!\!\!\!\bullet\; \mathbb{Q}\pi(X^{K_1}),$$

  where $\theta : G/K_1 \to\!\!\!\!\bullet\; G/L_1$.

  Note: The vertical double functors are invertible.

Double Category version of Orbit Category

- Double cells:

$$
\begin{array}{ccc}
\mathbb{Q}\pi(X^{K_1}) & \xleftarrow{\mathcal{F}a} & \mathbb{Q}\pi(X^{K_2}) \\
\mathcal{F}\theta_1 \uparrow & \mathcal{F}\xi & \uparrow \mathcal{F}\theta_2 \\
\mathbb{Q}\pi(X^{L_1}) & \xleftarrow[\mathcal{F}b]{} & \mathbb{Q}\pi(X^{L_2})
\end{array}
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is a vertical transformation,

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  is a vertical transformation,

  - For every $x \in X^{L_2}$, $\mathcal{F}\xi_x$ is a path in X^{K_1} from $a\theta_2(1)^{-1}x$ to $\theta_1(1)^{-1}bx$.

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  is a vertical transformation,

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  - For every horizontal arrow $u : x_1 \to x_2$ in X^{L_2}, $\mathcal{F}\xi_u$ is a path homotopy in X^{K_1} between $a\theta_2(1)^{-1}u * \mathcal{F}\xi_{x_2}$ and $\mathcal{F}\xi_{x_1} * \theta_1(1)^{-1}bu$.

Double Category version of Orbit Category

The tom Dieck fundamental double groupoid is obtained as a quotient of a categorical Grothendieck construction

$$\mathbb{P}^2_G(X) = \int_{\mathcal{O}_G} \mathcal{F}.$$

- Objects: $(G/H, x)$ where $x \in X^H$;
- Horizontal arrows:

$$(a, u) : (G/K_1, x_1) \to (G/K_2, x_2),$$

where $a : G/K_1 \to G/K_2$ is a G-equivariant map in $\mathbb{O}^2_G$ and u is a path in X^{K_1} from x_1 to ax_2.

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  where $a : G/K_1 \to G/K_2$ is a G-equivariant map in $\mathbb{O}^2_G$ and u is a path in X^{K_1} from x_1 to ax_2.
- Vertical arrows:

$$(\theta_1, \alpha_1) : (G/K_1, x_1) \mathrel{-\!\!\bullet\!\!\to} (G/L_1, y_1),$$

  where $\theta_1 : G/K_1 \mathrel{-\!\!\bullet\!\!\to} G/L_1$ is a vertical arrow in $\mathbb{O}^2_G$ and α_1 is a path in X^{K_1} from x_1 to $\theta_1(1)^{-1}y_1$.

Double Category version of Orbit Category

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(G/K_1, x_1) & \xrightarrow{(a,u)} & (G/K_2, x_2) \\
(\theta_1, \alpha_1) \big\downarrow & (\xi, \Phi) & \big\downarrow (\theta_2, \alpha_2) \\
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\end{array}
\qquad
\begin{array}{ccc}
x_1 & \xrightarrow{u} & ax_2 \\
 & & \big\downarrow {\scriptstyle a\alpha_2} \\
{\scriptstyle \alpha_1} \big\downarrow & \Phi & a\theta_2(1)^{-1}y_2 \\
 & & \big\downarrow {\scriptstyle \mathcal{F}\xi_{y_2}} \\
\theta_1(1)^{-1}y_1 & \xrightarrow[\theta_1(1)^{-1}v]{} & \theta_1(1)^{-1}by_2
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\begin{array}{ccccc}
x_1 & \xrightarrow{u} & ax_2 & \xrightarrow{a\alpha_2} & a\theta_2(1)^{-1}y_2 \\
{[id_x]} \big| & & & & \big\downarrow \mathcal{F}\xi_{y_2} \\
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Higher Topological Complexity

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example

Consider the action of $G = \mathbb{S}^1 \times \mathbb{Z}_2 \subset \mathbb{C} \times \mathbb{Z}_2$ on a cylinder $C = \mathbb{S}^1 \times [-1, 1]$ defined by

$$G \times C \to C$$
$$(\alpha, \beta) \cdot (t, x) = (\alpha t, \beta x)$$

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- For any point $x \in C$ on the central circle of C, we have $G_x = K = \{0\} \times \mathbb{Z}_2$.

$$X^K = \mathbb{S}^1 \times \{0\}.$$

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- For any point $x \in C$ on the central circle of C, we have $G_x = K = \{0\} \times \mathbb{Z}_2$.

$$X^K = \mathbb{S}^1 \times \{0\}.$$

- For any other point $y \in C$, we have $G_y = E$ is trivial.

$$X^E = X.$$

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Thank you!