Lecture 5 - Monads, Part II

Last time - Monads on a category $\mathcal{C}$

$$(T: \mathcal{C} \to \mathcal{C}, \mu: T^2 \to T, \eta: \mathrm{id} \to T)$$

satisfying several equations.

One more example:

Let M be a monoid. Let $e \in M$ be the unit for M.

$T: \mathrm{Set} \to \mathrm{Set}$ $\quad T(X) = M \times X$

$\mu: T^2 \to T$ $\quad \mu_X: T^2 X \to TX$

$: M \times M \times X \to M \times X$

$(m_1, m_2, x) \mapsto (m_1 m_2, x)$

$\eta: \mathrm{id} \to T$ $\quad \eta_X: X \to TX$

$x \mapsto (e, x)$

equations are straightforward.

This is the free M-Set monad.

What is its universal property?

Let Y be any M-Set. So I have a map $\rho: M \times Y \to Y$

Let $f: X \to Y$ be any function. (Remember X is just a set, so there is nothing for f to preserve.)

$\exists$ a unique map of M-Sets

$\hat{f}: M \times X \to Y$ s.t.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "f"'] & M \times X \arrow[d, "\hat{f}"] \\
 & Y
\end{tikzcd}$$

What is $\hat{f}$? $\hat{f}(m,x) = mf(x)$

Understand this example.

Last time

Thm: Suppose I have an adjunction

$$F : \mathcal{C} \to \mathcal{D} \qquad G : \mathcal{D} \to \mathcal{C}$$

with natural transformations

$$\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow G \circ F$$

$$\varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

$(T = G \circ F,\ G(\varepsilon_F), \eta)$ is a monad.

(Make sure you understand what $G(\varepsilon_F)$ is)

The proof is a straight diagram check

Is the converse correct? Yes, in several different ways.

Solution #1 Kleisli Category

denoted $\mathcal{C}_T$.

Objects are same as in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}_T}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,TY)$$

Use structure of the monad to show this is a category. In particular, how do you compose 2 arrows in the Kleisli category?

$$F:\mathcal{C} \to \mathcal{C}_T \quad G:\mathcal{C}_T \to \mathcal{C}$$

$$F(X) = X \quad F(f:X\to Y) = X \xrightarrow{f} Y \xrightarrow{\eta} TY$$

$$G(X) = TX \quad G(g: X \to TY) =$$

$$TX \xrightarrow{T(g)} T^2Y \xrightarrow{\mu} TY$$

But in general, there are many adjunctions which yield the same monad.

Eilenberg-Moore Algebras or T-algebras

Let (T, μ, η) be a monad on $\mathcal{C}$. A <u>T-algebra</u> is a pair $(A, h : TA \to A)$ such that

$$\begin{tikzcd} T^2 A \arrow[r, "Th"] \arrow[d, "\mu_A"'] & TA \arrow[d, "h"] \\ TA \arrow[r, "h"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal] & TA \arrow[d, "h"] \\ & A \end{tikzcd}$$

A morphism of T-algebras is a map $f : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "h"'] & TB \arrow[d, "h'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Easy to check this is a category. Call it T-Alg or ~~$\mathcal{C}^*$~~ $\mathcal{C}^T$

Now to build an adjunction

$F: \mathcal{C} \to \mathcal{C}^T \quad G: \mathcal{C}^T \to \mathcal{C}$

$G((A,h)) \mapsto A$ It's a forgetful functor

$F(X) = (TX, \mu)$. Easy to see this is a T-algebra, and the rest is straightforward verification.

Examples of T-algebras.

i) Let P be a partially ordered set, viewed as a category. So there is an arrow $x \to y$ iff $x \leq y$. A monad on P is an order-preserving map $T: P \to P$ s.t. $\forall x \; x \leq Tx$, and $T^2x \leq Tx$. These imply that $T^2x = Tx$.

This is a <u>closure operator</u> on P.

So for example, let X be a topological space. Let $P = \mathcal{P}(X)$. Then

$$U \mapsto \overline{U}$$

is an example.

A T-algebra in this case is an element $x \in P$ s.t. $T(x) \leq x$ So $T(x) = x$.

These are precisely the closed elements. In our topological example, these are precisely the closed sets.

EX2: Let M be a monoid

Let $T: Set \to Set$ be defined by

$$T(X) = M \times X$$

As already seen, it's a monad.

What's a T-algebra?

A function $h: M \times X \to X$. If I denote $h(m,x)$ by $m \circ x$, The T-algebra equations become

$$m_1 \circ (m_2 \circ x) = (m_1 m_2) \circ x$$

$$1 \circ x = x$$

So we get the familiar notion of

M-Set. If G is a group, we get the standard notion of G-Set.

<u>Ex3</u> We have the free monoid monad.
$T(X)$ is the set of all words built from elements of X. Let $h: TX \to X$ be a T-algebra.

If $x, y \in X$, define $x \cdot y = h(xy) \in X$.
Define $e \in X$ by $e = h(\varepsilon)$
empty list.

The T-algebra equations say that these operations make X a monoid, and a morphism of T-algebras is a monoid homomorphism.

In fact, in this case.

Thm: If T is the free monoid monad, then the category $\mathcal{C}^T$ of T-algebras is equivalent to the category of all monoids

Similarly, if T is the free M-Set monad, then the category of T-algebras is equivalent to the category of M-Sets.

This is a common occurence. You have some category of algebraic structures $\mathcal{C}$ and a forgetful functor

$$G : \mathcal{C} \to \mathrm{Set}$$

which has a left adjoint, F and this induces a monad on Set. We say that G is <u>monadic</u> or <u>tripleable</u> if $\mathcal{C}$ is equivalent to the category of T-algebras.

We also say $\mathcal{C}$ is <u>monadic over Sets</u>.

Note: G doesn't have to go to Sets, but it usually does.

There are cases where monadicity fails.

Nonexample. Let $\mathcal{C}$ be the category of Topological spaces & continuous maps

$$G : \mathcal{C} \to \text{Set}$$

is the forgetful functor.
A left adjoint to G is given by the discrete topology. The resulting monad is just the identity, and the category of Sets.

So the intuition is that if $\mathcal{C}$ is monadic over sets, it is algebraic.

In particular, every algebraic theory is monadic over Sets, so

groups, monoids, rings, G-Sets, etc

are all monadic.

Thm (Manes) The category of compact hausdorff spaces is monadic over Sets.

In general, proving monadicity is quite difficult. There are theorems by Jon Beck, which can help.

See Ch VI.7 of MacLane for more details.

In general, there can be many adjoint pairs that induce the same monad.

The set of all such adjoint pairs inducing a fixed T can be formed into a category.

Thm (See Ch VI.5 of MacLane)

The Kleisli construction is initial in this category and the Eilenberg-Moore construction is terminal.

<u>One More Example</u>

p 142 of Mac Lane.

A <u>complete sup-lattice</u> is a partially ordered set such that every subset has a supremum, i.e. a least upper bound

Notes: 1) If L is a complete sup lattice, it has a top element

$$\top = \bigvee L$$

and

$$\bot = \bigvee \emptyset$$

2) If L has all sups, it also has all inf's. So why call them sup-lattices?

Defn: A <u>morphism of complete sup-lattices</u>, $f: M \to N$ is a map preserving all sups.

NB (This does not imply it preserves all infs)

Reminder $\mathcal{P}: \mathrm{Set} \to \mathrm{Set}$, the power set monad.

$\mu: \mathcal{P}^2(X) \to \mathcal{P}(X)$ takes the union

$$\eta: X \to \mathcal{P}(X)$$
$$x \mapsto \{x\}$$

Thm: Every $\mathcal{P}$-algebra is a complete sup lattice. Define an order on $(X, h: \mathcal{P}X \to X)$ by $x \le y$ iff

$$h(\{x,y\}) = y.$$

This gives an order, and with this order

$$h(U \subseteq X) \text{ becomes the}$$

sup of U.

Conversely, every complete sup lattice so arises.

Thus, sup-lattices are monadic over Set.

Finally, everything we've done dualizes.

Comonads or Cotriples

Comonad $\quad C : \mathcal{C} \to \mathcal{C}$

$$\delta : C \to C^2$$

$$\varepsilon : C \to \mathrm{id}$$

CoKleisli Category $\mathcal{C}_C$

$$\mathrm{Hom}_{\mathcal{C}_C}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(CX, Y)$$

Eilenberg-Moore Coalgebras

$$(X, p) \quad p : X \to CX$$

etc

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CoKleisli Category $\mathcal{C}_C$

$$\mathrm{Hom}_{\mathcal{C}_C}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(CX,Y)$$

Eilenberg-Moore Coalgebras

$(X, \rho) \quad \rho : X \to CX$

etc