Inner autoequivalences of monoidal categories

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Punchline

Recall: an automorphism of a group G is inner if it is of the form $x \mapsto gxg^{-1}$ for some $g \in G$.

Theorem (Informal)

Inner autoequivalences of a monoidal category $\mathcal{C}$ are of the form $A \otimes - \otimes A^{-1}$ where $A \otimes A^{-1} \cong I \cong A^{-1} \otimes A$.

Perhaps more importantly:

- there is a general theory of inner automorphisms in a category (known)
- and of inner autoequivalences in a 2-category (new)

Inner automorphisms

Given $g \in G$, we have more than the function $\alpha_{\mathrm{id}} \colon G \to G$ given by conjugation with g:

- For any $f \colon G \to H$, we get a function $\alpha_f \colon H \to H$ given by conjugation with $f(g)$
- Moreover, for any $h \colon H \to H'$ the square

$$\begin{array}{ccc} H & \xrightarrow{\alpha_f} & H \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H' & \xrightarrow[\alpha_{hf}]{} & H' \end{array}$$

  commutes.

In other words, g induces a natural automorphism of the projection $G/\mathcal{G}rp \to \mathcal{G}rp$. A theorem by Bergman shows that any such natural automorphism is induced in this way

Inner automorphisms

Definition

The (covariant) isotropy group of $\mathcal{C}$ is the functor $\mathcal{Z}\colon \mathcal{C} \to \mathcal{G}rp$ defined on objects by sending A to $\mathrm{Aut}(P_A\colon A/\mathcal{C} \to \mathcal{C})$. For a morphism $g\colon A \to B$ the homomorphism $\mathcal{Z}(g)$ is defined by whiskering with $g^*\colon B/\mathcal{C} \to A/\mathcal{C}$.

Explicitly, an element $\alpha \in \mathcal{Z}(A)$ consists of an automorphism $\alpha_f\colon B \to B$ for every $f\colon A \to B$, such that

$$\begin{array}{ccc} B & \xrightarrow{\alpha_f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[\alpha_{hf}]{} & C \end{array}$$

commutes for any h, and for $g\colon A \to B$, $\mathcal{Z}(g)(\alpha) \in \mathcal{Z}(B)$ is defined by $(\mathcal{Z}(g)(\alpha))_h = \alpha_{hg}$.

What is known

For algebraic theories (and essentially algebraic theories) the isotropy group coincides with the group of *definable* automorphisms: for precise details, see '*Polymorphic Automorphisms and the Picard Group*'
Hofstra, Parker & Scott, FSCD 2021.

Using this, they show that for the 1-category $\mathcal{StrMonCat}$ of strict monoidal categories and strict monoidal functors, the isotropy group of $\mathcal{C} \in \mathcal{StrMonCat}$ coincides with the group of strictly invertible objects of $\mathcal{C}$.

A guess

This leads to a natural guess: for general monoidal categories, one should get weakly invertible elements, i.e. objects A for which an object A^{-1} exists with $A \otimes A^{-1} \cong I \cong A^{-1} \otimes A$.

This requires two things:

- When A is weakly invertible, the functor $A \otimes - \otimes A^{-1}$ is a strong monoidal auto*equivalence*, so we need a 2-d version of the above
- Figuring out how to compute this, perhaps without relying on "definability" and other logical/syntactic tools

Bicategorical isotropy

Definition

For a bicategory $\mathcal{C}$, define a pseudofunctor $\mathcal{Z}$ as follows:

- send an object A to the monoidal category whose objects are pseudonatural autoequivalences of $(A/\mathcal{C})_{ps} \to \mathcal{C}$ and (invertible?) modifications between them
- on 1-cells by the same whiskering as before, resulting in a strong monoidal functor
- 2-cells induce monoidal natural transformations

There's a catch in the definition:

- If we take only invertible modifications, then $\mathcal{Z}(A)$ is a (weak) 2-group. However, then we have to restict $\mathcal{C}$ to invertible 2-cells.
- Take all modifications, and replace $(A/\mathcal{C})_{ps}$ by $(A/\mathcal{C})_{lax}$: then $\mathcal{Z}$ is a pseudofunctor for all of $\mathcal{C}$.

For this talk, the choice doesn't affect the results that much.

Density, recap

Recall:

Proposition

The following conditions for a functor $F\colon \mathcal{C} \to \mathcal{D}$ *are equivalent:*

(i) *the restricted Yoneda embedding* $\mathcal{D} \xrightarrow{よ} [\mathcal{D}^{\mathrm{op}}, \mathcal{S}et] \xrightarrow{[F,\mathrm{id}]} [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$ *is fully faithful*

(ii) *for any* $\mathcal{E}$*, the restriction of* $[\mathcal{D}, \mathcal{E}] \xrightarrow{[F,\mathrm{id}]} [\mathcal{C}, \mathcal{E}]$ *to cocontinuous functors* $\mathcal{D} \to \mathcal{E}$ *is a fully faithful.*

A functor with these properties is called *dense*.

Density as a tool

Assume $\mathcal{C}$ has binary coproducts, so that each projection $P_A\colon A/\mathcal{C} \to \mathcal{C}$ has a left adjoint L_A sending X to $i_A\colon A \to A + X$. Let $F\colon \mathcal{D} \to \mathcal{C}$ be dense and define a functor $\mathcal{L}\colon \mathcal{C} \to \mathcal{Grp}$ by sending $A \in \mathcal{C}$ to $\mathrm{Aut}(L_A F)$.

Theorem

There is a natural isomorphism $\mathcal{Z} \cong \mathcal{L}$. In particular, if $\mathcal{D}$ can be chosen to be small, then $\mathcal{Z}$ lands in small groups.

Proof.

Take mates and use the fact that left adjoints are cocontinuous. □

Density as a tool

- The free group on two generators is dense in $\mathcal{G}rp$
- Taking the full subcategory spanned by (i) the free monoidal category on two objects and (ii) the free monoidal category on a morphism, is dense in the category strict monoidal categories and strict monoidal functors
- The walking isomorphism is dense in the 1-category $\mathcal{G}rpd$

This explains why certain coproducts show up in previous computations of isotropy groups.

Bicategorical density

A straightforward extension shows:

Proposition

The following conditions for a pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ are equivalent:

(i) *the restricted Yoneda embedding $\mathcal{D} \xrightarrow{よ} [\mathcal{D}^{\mathrm{op}}, \mathcal{C}at]_{ps} \xrightarrow{[F,\mathrm{id}]} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]_{ps}$ is a local equivalence.*

(ii) *for any $\mathcal{E}$, the restriction of $[\mathcal{D}, \mathcal{E}]_{ps} \xrightarrow{[F,\mathrm{id}]} [\mathcal{C}, \mathcal{E}]_{ps}$ to cocontinuous pseudofunctors $\mathcal{D} \to \mathcal{E}$ is a local equivalence.*

We will call such pseudofunctors *dense*.

Density as a tool

Assume a bicategory $\mathcal{C}$ has binary (bi)coproducts, so that each projection $P_A\colon A/\mathcal{C} \to \mathcal{C}$ has a left (bi)adjoint L_A sending X to $i_A\colon A \to A + X$. Let $F\colon \mathcal{D} \to \mathcal{C}$ be dense and define a pseudofunctor $\mathcal{L}$ by sending $A \in \mathcal{C}$ to $\mathrm{Aut}(L_A F)$.

Theorem

There is a pseudonatural equivalence $\mathcal{Z} \simeq \mathcal{L}$.

Example: 2-category of groupoids

The trivial groupoid 1 is dense in the 2-category $\mathcal{G}rpd$ of groupoids.

For a groupoid G, a pseudonatural autoequivalence of $1 \to \mathcal{G}rpd \to G/\mathcal{G}rpd$ is determined by an autoequivalence of $G + 1$ fixing G (up to iso).

Such an autoequivalence has to fix 1, so all such autoequivalences are isomorphic. Thus $\mathcal{Z}(G)$ is equivalent to the trivial 2-group.

Monoidal cats

Let's work in the 2-category of *strict* monoidal categories, strong monoidal functors and monoidal natural *isomorphisms*
Let $\mathcal{F}$ be the free monoidal category on two objects X, Y. The subcategory it spans is dense.
Then an element of $\mathcal{Z}(\mathcal{C})$ is more or less determined by an autoequivalence $\alpha\colon \mathcal{C} + \mathcal{F} \to \mathcal{C} + \mathcal{F}$ fixing $\mathcal{C}$.

- Pseudonaturality at the map $X \mapsto Y, Y \mapsto X$ implies that α is determined by a single word $w(-)$ in objects of $\mathcal{C}$ with a free variable, where $\alpha(X) \cong w(X)$ and $\alpha(Y) \cong w(Y)$.
- Pseudonaturality at $X, Y \mapsto X \otimes Y$ implies that $w(X \otimes Y) \cong w(X) \otimes w(Y)$. On the RHS, each X is to the left of each Y, so that $w(-)$ has to use the free variable at most once - and at least once to be an equivalence.
- Thus $w(X) = A \otimes X \otimes B$ for some $A, B \in \mathcal{C}$. Strong monoidality implies $B \otimes A \cong I$, and pseudonaturality at $X, Y \mapsto I$ implies that $A \otimes B \cong I$.

Monoidal cats

Filling some details in, we see that $\mathcal{Z}$ is pseudonaturally equivalent to the functor sending $\mathcal{C}$ to the Picard 2-group of $\mathcal{C}$, i.e. the 2-group on weakly invertible objects and isomorphisms between them.

Working with all monoidal natural transformations, you get weakly invertible objects and all maps between them.

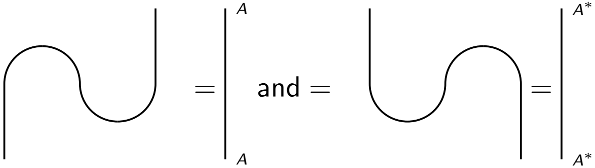
Work in progress: monoidal cats and dual objects

Instead of inner autoequivalences, one could study inner endomorphisms. In 2-d, there's an intermediate option: inner ambidextrous adjunctions.

Recap:

- A Frobenius monoidal functor is a functor that is lax monoidal and oplax monoidal in a compatible manner
- In a monoidal category, a left dual for an object A consists of an object A^* and maps

such that



We say that A^* is a an ambidextrous dual of A if it is both a left and right dual

Work in progress: monoidal cats and dual objects

Let $\mathcal{Moncat}_{Frob}$ stand for the 2-category of monoidal categories, Frobenius monoidal functors and monoidal natural transformations.

Theorem (Tentative)

Every lax natural ambiadjunction $\mathcal{C}/\mathcal{Moncat}_{Frob} \to \mathcal{Moncat}_{Frob}$ is of the form $A \otimes - \otimes A^*$ where A^* is an ambidextrous dual of $A \in \mathcal{C}$.

Questions...

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