Introduction to Double Categories, Part 1

Susan Niefield

Union College
Schenectady, NY

February 4, 2021

Double Categories

A **double category** $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (\mathrm{id}^\bullet : \mathbb{D}_0 \to \mathbb{D}_1)$$

i.e., $s \circ \mathrm{id}^\bullet = t \circ \mathrm{id}^\bullet = \mathrm{id}_{\mathbb{D}_0}$, $s \circ \odot = s$, $t \circ \odot = t$, and $\odot$ is associative with unit $\mathrm{id}^\bullet$, up to coherent isomorphism.

Objects X of $\mathbb{D}_0$: **objects** of $\mathbb{D}$

Morphisms $f : X \longrightarrow Y$ of $\mathbb{D}_0$: **horizontal** or **tight morphisms** of $\mathbb{D}$

Objects $v : X_s \to X_t$ of $\mathbb{D}_1$: **vertical** or **loose morphism** of $\mathbb{D}$

Morphisms $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v \downarrow & \varphi & \downarrow w \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$ of $\mathbb{D}_1$: **cells** of $\mathbb{D}$

Why Double Categories?

Two kinds of morphisms in one category

- $\mathcal{T}(\mathbb{D})$: 2-category of objects, tight morphisms, and cells
- $\mathcal{L}(\mathbb{D})$: bicategory of objects, loose morphisms, and cells

Introduced by Ehresmann (1963)

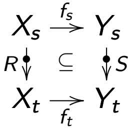
Papers by Grandis and Paré (1999 -)

Papers by Shulman (2008 -)

Many others

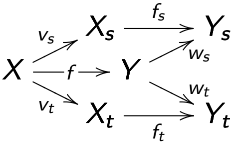
Set-Like Double Categories: $\mathbb{D}_0 = \mathbf{Sets}$

$\mathbb{R}\mathrm{el}_1$:



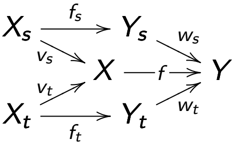
$(x_s, x_t) \in R \Rightarrow (f_s x_s, f_t x_t) \in S$

$\mathbb{S}\mathrm{pan}_1$:



$\odot$ via pullback

$\mathbb{C}\mathrm{ospan}_1$:



$\odot$ via pushout

Can define $\mathbb{S}\mathrm{pan}$ (resp, $\mathbb{C}\mathrm{ospan}$) for $\mathbb{D}_0$ with pullbacks (resp, pushouts).

Space-Like Double Categories

$$\mathbb{T}\mathrm{op}: \text{ top spaces } X, \quad \underset{\text{cont maps}}{X \xrightarrow{f} Y}, \quad \underset{\text{pres } \wedge, \top}{\dfrac{X_s \overset{}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} X_t}{\mathcal{O}(X_s) \xrightarrow[v]{} \mathcal{O}(X_t)}}, \quad \begin{array}{ccc} \mathcal{O}(X_s) & \xrightarrow{\mathcal{O}(f_s)} & \mathcal{O}(Y_s) \\ {\scriptstyle v}\big\downarrow\!\bullet & \supseteq & \bullet\!\big\downarrow{\scriptstyle w} \\ \mathcal{O}(X_t) & \xrightarrow[\mathcal{O}(f_t)]{} & \mathcal{O}(Y_t) \end{array}$$

$$\mathbb{T}\mathrm{opos}: \text{ toposes } \mathcal{X}, \quad \underset{\text{geom morph}}{\mathcal{X} \xrightarrow{f} \mathcal{Y}}, \quad \underset{\text{lex}}{\mathcal{X}_s \xrightarrow{v} \mathcal{X}_t}, \quad \begin{array}{ccc} \mathcal{X}_s & \xrightarrow{f_s} & \mathcal{Y}_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \leftarrow & \bullet\!\big\downarrow{\scriptstyle w} \\ \mathcal{X}_t & \xrightarrow[f_t]{} & \mathcal{Y}_t \end{array}$$

$$\mathbb{L}\mathrm{oc}: \text{ locales } X, \quad \underset{\text{locale maps}}{X \xrightarrow{f} Y}, \quad \underset{\text{pres } \wedge, \top}{X_s \xrightarrow{v} X_t}, \quad \begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \geq & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

$$\mathbb{Q}\mathrm{uant}^{\mathrm{op}}: \text{ quantales } Q, \quad \underset{\substack{\exists f^* \vdash f \\ f^* \text{ pres } \cdot, e}}{Q \xrightarrow{f} R}, \quad \underset{\substack{v(x_s)v(y_s) \leq v(x_s y_s) \\ e_t \leq v(e_s)}}{Q_s \xrightarrow{v} Q_t}, \quad \begin{array}{ccc} Q_s & \xrightarrow{f_s} & R_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \geq & \bullet\!\big\downarrow{\scriptstyle w} \\ Q_t & \xrightarrow[f_t]{} & R_t \end{array}$$

Ring-Like Double Categories

$\mathbb{R}\mathrm{ing}$: rings R, $\underset{\text{homoms}}{R \xrightarrow{f} S}$, $\underset{\text{bimodules}}{R_s \overset{M}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} R_t}$, $\begin{array}{ccc} R_s & \xrightarrow{f_s} & S_s \\ {\scriptstyle M}\big\downarrow & \overset{\varphi}{\Rightarrow} & \big\downarrow{\scriptstyle N} \\ R_t & \xrightarrow[f_t]{} & S_t \end{array}$ $\quad \odot$ via $\otimes$

$\mathbb{Q}\mathrm{uant}$: quantales Q, $\underset{\text{homoms}}{Q \xrightarrow{f} R}$, $\underset{\text{bimodules}}{Q_s \overset{M}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Q_t}$, $\begin{array}{ccc} Q_s & \xrightarrow{f_s} & R_s \\ {\scriptstyle M}\big\downarrow & \overset{\varphi}{\Rightarrow} & \big\downarrow{\scriptstyle N} \\ Q_t & \xrightarrow[f_t]{} & R_t \end{array}$ $\quad \odot$ via $\otimes$

This generalizes to the double category $\mathcal{V}$-$\mathbb{B}\mathrm{imod}$ of monoids, homomorphisms, and bimodules in any symmetric monoidal category $\mathcal{V}$.

Cat-Like Double Categories

$\mathbb{C}\mathrm{at}$: categories X, $\underset{\text{functors}}{X \longrightarrow Y}$, $\underset{\text{profunctors}}{\dfrac{X_s \overset{\bullet}{\longrightarrow} X_t}{X_s^{\mathrm{op}} \times X_t \underset{v}{\longrightarrow} \mathrm{Sets}}}$, $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\,\big\downarrow\!\bullet & \varphi & \bullet\!\big\downarrow\,{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$ $\odot$ via coends

$$v(x_s, x_t) \underset{\varphi}{\longrightarrow} w(f_s x_s, f_t x_t)$$

$\mathbb{P}\mathrm{os}$: posets X, $\underset{\text{monotone}}{X \longrightarrow Y}$, $\underset{\text{order ideals}}{X_s \overset{\bullet}{\longrightarrow} X_t}$, $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\,\big\downarrow\!\bullet & \subseteq & \bullet\!\big\downarrow\,{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$

One can define the double category $\mathcal{V}$-$\mathbb{C}\mathrm{at}$ of $\mathcal{V}$-enriched categories, $\mathcal{V}$-functors, and $\mathcal{V}$-profunctors for any symmetric monoidal $\mathcal{V}$, giving:

$\mathbb{Q}\mathrm{tld}$: quantaloids, when $\mathcal{V} = \mathrm{Sup}$, i.e., sup-lattices

$\mathbb{R}\mathrm{ngd}$: ringoids, when $\mathcal{V} = \mathrm{Ab}$, i.e., abelian groups

$\mathbb{O}\mathrm{rd}$: preorders, when $\mathcal{V} = \mathbb{2}$

Fibrant Double Categories (aka Framed Bicategories)

A double category $\mathbb{D}$ is a called **fibrant** if every $X \xrightarrow{f} Y$ has a **companion**, i.e., a vertical morphism $X \overset{f_*}{\to\!\!\!\!\!\bullet\,} Y$ and cells

$$
\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
{\scriptstyle \mathrm{id}^\bullet_X}\downarrow & \eta & \downarrow{\scriptstyle f_*} \\
X & \xrightarrow[f]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle f_*}\downarrow & \varepsilon & \downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\
Y & \xrightarrow[\mathrm{id}_Y]{} & Y
\end{array}
$$

whose horizontal and vertical compositions are identities, and a **conjoint**, i.e., a vertical morphism $Y \overset{f^*}{\to\!\!\!\!\!\bullet\,} X$ and cells

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \mathrm{id}^\bullet_X}\downarrow & \alpha & \downarrow{\scriptstyle f^*} \\
X & \xrightarrow[\mathrm{id}_X]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{\mathrm{id}_Y} & Y \\
{\scriptstyle f^*}\downarrow & \beta & \downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

whose horizontal and vertical compositions are identities.

Examples

All the examples are fibrant, i.e., given $f\colon X \longrightarrow Y$,

$\mathbb{R}\mathrm{el}$: $f_* = \{\,(x,y) \mid y = fx\}$ and $f^* = \{\,(y,x) \mid y = fx\}$

$\mathbb{C}\mathrm{ospan}, \mathbb{S}\mathrm{pan}$: f as one leg and the identity the other

$\mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}, \mathbb{Q}\mathrm{uant}^{\mathrm{op}}$: direct/inverse images

$\mathbb{P}\mathrm{os}$: $f_* = \{\,(x,y) \mid fx \leq y\}$ and $f^* = \{\,(y,x) \mid y \leq fx\}$

$\mathcal{V}$-$\mathbb{C}\mathrm{at}$: $f_*(x,y) = Y(fx,y)$ and $f^*(y,x) = Y(y,fx)$

$\mathbb{B}\mathrm{imod}(\mathcal{V})$: Y as an (X,Y)- and (Y,X)-bimodule via f

Fibrant Double Categories, $\mathbb{S}\mathrm{pan}$, and $\mathbb{C}\mathrm{ospan}$

A **lax functor** $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ consists of functors $F_0\colon \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ and $F_1\colon \mathbb{C}_1 \longrightarrow \mathbb{D}_1$ (both denoted F) compatible* with s and t, and cells

$$\mathrm{id}^\bullet_{FX} \xrightarrow{\rho_X} F(\mathrm{id}^\bullet_X) \quad \text{and} \quad Fv \odot Fv' \xrightarrow{\rho_{v,v'}} F(v \odot v')$$

with naturality/coherence conditions. F is called **normal**, if ρ_X is invertible, and F is called **pseudo**, if ρ_X and $\rho_{v,v'}$ are invertible.

*compatible:

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow & \varphi & \downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array} \quad \mapsto \quad \begin{array}{ccc} FX_s & \xrightarrow{Ff_s} & FY_s \\ {\scriptstyle Fv}\downarrow & F\varphi & \downarrow{\scriptstyle Fw} \\ FX_t & \xrightarrow[Ff_t]{} & FY_t \end{array}$$

Fibrant Double Categories, $\mathbb{S}\text{pan}$, and $\mathbb{C}\text{ospan}$, cont.

Proposition (N, TAC 2012)

Suppose $\mathbb{D}_0$ has pushouts. Then $\mathbb{D}$ is fibrant if and only if $\mathrm{id}_{\mathbb{D}_0}$ extends to a normal lax functor $F\colon \mathbb{C}\text{ospan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$.

Proof.

$(\Rightarrow)$ $F(X_s \xrightarrow{v_s} X \xleftarrow{v_t} X_t) = X_s \xrightarrow{{v_s}_*} X \xrightarrow{{v_t}^*} X_t$

$(\Leftarrow)$ $f_* = F(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y)$ and $f^* = F(Y \xrightarrow{\mathrm{id}_Y} Y \xleftarrow{f} X)$ □

Applying the proposition to $\mathbb{D}_0^{\mathrm{op}}$ gives:

Corollary

Suppose $\mathbb{D}_0$ has pullbacks. Then $\mathbb{D}$ is fibrant if and only if $\mathrm{id}_{\mathbb{D}_0}$ extends to a normal lax functor $\mathbb{S}\text{pan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$.

Remark: These results apply to all the examples here.

Cartesian Double Categories

There is a 2-category **LxDbl** with double categories as 0-cells, lax functors as 1-cells, and suitable* 2-cells. It has a sub-2-category **PsDbl** whose 1-cells are pseudo functors. Both have finite 2-products with $\mathbb{C} \times \mathbb{D}$ defined point-wise and terminal object $\mathbb{1}$.

*suitable = horizontal transformations, in the sense of Grandis/Paré

Following Aleiferi, we say $\mathbb{D}$ is **pre-cartesian** if $\Delta \colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $! \colon \mathbb{D} \to \mathbb{1}$ have right adjoints in **LxDbl**, which we denote $\times$ and 1; and $\mathbb{D}$ is **cartesian** if $\times$ and 1 are pseudo, i.e., adjoints in **PsDbl**.

Can show:

$\mathbb{R}\mathrm{el}, \mathbb{S}\mathrm{pan}, \mathbb{C}\mathrm{ospan}, \mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}$ are cartesian

$\mathcal{V}$-$\mathbb{C}\mathrm{at}$, $\mathbb{B}\mathrm{imod}(\mathcal{V})$ are pre-cartesian, for $\mathcal{V}$ cartesian

$\mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}$ are pre-cartesian

References

- E. Aleiferi, Cartesian Double Categories with an Emphasis on Characterizing Spans, https://arxiv.org/abs/1809.06940.
- C. Ehresmann. Catégories structurées, Ann. Sci. École Norm. Sup. 80 (1963), 349–426.
- M. Grandis and R. Paré, Limits in double categories, Cahiers 40 (1999),162–220.
- S. Niefield, Span, cospan, and other double categories, TAC 26, (2012), 729–742
- R. Paré, Yoneda theory for double categories, TAC 25 (2011), 436–489.
- M. Shulman, Framed bicategories and monoidal fibrations, TAC 20 (2008), 650–738.