Softness of Hypercoherences and MALL Full Completeness

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Abstract

We prove a full completeness theorem for multiplicative-additive linear logic (i.e. MALL) using a double gluing construction applied to Ehrhard's $*$-autonomous category of hypercoherences. This is the first non-game-theoretic full completeness theorem for this fragment. Our main result is that every dinatural transformation between definable functors arises from the denotation of a cut-free MALL proof.

Our proof consists of three steps. We show:

- Dinatural transformations on this category satisfy Joyal's softness property for products and coproducts.
- Softness, together with multiplicative full completeness, guarantees that every dinatural transformation corresponds to a Girard MALL proof-structure.
- The proof-structure associated to any dinatural transformation is a MALL proof-net, hence a denotation of a proof. This last step involves a detailed study of cycles in additive proof structures.

Contents

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1 Introduction

1.1 History of Full Completeness

Linear Logic [17] first arose from Girard's semantic investigations of the category **Coh** of coherent spaces with stable maps, a simplification of Scott domains. As Girard [17] says: "Linear logic first appeared as a kind of linear algebra built on coherent spaces . . . ." Later Thomas Ehrhard [15] established a substantial refinement of **Coh**, the category **HCoh** of hypercoherences. Hypercoherences arose from the Bucciarelli-Ehrhard investigations [11] of sequentiality, using strong stability in qualitative domains endowed with coherences. Sequentiality itself is an important issue in programming language semantics, closely related to the so-called *full abstraction problem* [12]. The key property of the hypercoherence model is that it eliminates certain well-known non-sequential boolean functions, namely n-ary analogs of Berry's "Gustave" functions, which are extensions of the familiar *parallel-or*.

The logical counterpart to full abstraction is full completeness (the terminology comes from Abramsky-Jagadeesan [1]). Full completeness theorems are completeness theorems at the level of *proofs*, rather than *provability*. More precisely, given a logic $\mathcal{L}$, we say a (categorical) model $\mathcal{M}$ for $\mathcal{L}$ is *fully complete* if in the unique $\mathcal{M}$-interpretation $[\![-]\!]$ of $\mathcal{L}$, every morphism $[\![A]\!] \xrightarrow{f} [\![B]\!] \in \mathcal{M}$ is the interpretation of a proof π of $A \vdash B$. At the level of categories, full completeness is a kind of *representation theorem*. If we identify $\mathcal{L}$ with an appropriately structured free category $\mathcal{F}$, then full completeness says the unique free functor $[\![-]\!]: \mathcal{F} \to \mathcal{M}$ is full. Of course it would be preferable if the unique interpretation functor $[\![-]\!]$ were fully faithful. This has been the case in our previous full completeness results for MLL [9, 10, 23, 24]. For the additives, this involves subtle problems concerning equality of proofs. This is discussed further in the Conclusion.

The first fully complete models for multiplicative fragments of linear logic were in Abramsky-Jagadeesan [1] for MLL + Mix and Hyland-Ong [29] for MLL, and were based on game semantics. More recently, variations of the categorical notion of *dinatural transformations* have been seen to provide a useful semantical framework for discussing full completeness. They were first proposed in [5] as a powerful functorial semantics for polymorphism, and later extended to intuitionistic logic [22] and linear logic [7]. Dinaturality provides a framework for imposing uniformity conditions on the interpretation in the model. Blute and Scott [9] proved that dinatural transformations over topological vector spaces provide a fully complete model for MLL + Mix . They also [10] extended their full completeness theorems to cyclic linear logic, by considering dinaturals invariant under (continuous) action of Hopf algebras on these vector spaces. Hamano [24] used Pontrjagin duality to extend the dinatural framework in [9, 10] to get full completeness for MLL. Recently, Abramsky and Melliès [2] announced a full completeness theorem for MALL, based on a dinatural framework over their notion of *concurrent games*.

In a different direction, Loader's thesis [33] contained a dinatural approach to full completeness. This work was generalized by Hyland and his student Tan [34], to certain $*$-autonomous *double gluing* categories $\mathbf{G}\mathcal{C}$. The construction arose from a generalization of Loader's linear logical predicates [33] in the case where the category $\mathcal{C}$ is the category **Rel** of sets and relations. More generally, this construction begins with any $*$-autonomous category $\mathcal{C}$ (i.e. a model of MLL) [6] and yields a new $*$-autonomous category $\mathbf{G}\mathcal{C}$ which is a better denotational model of proofs ("better" in that many unwanted morphisms are eliminated in the construction), see Section 2. For example, in most cases of interest, double gluing allows us to eliminate the Mix rule.

More fundamentally, double gluing is used in building fully complete MLL models [33, 34]. In the framework of Girard's coherent spaces, Tan [34] proved a full completeness theorem for the multiplicative fragment MLL+Mix, which states that every non-trivial dinatural transformation between MLL-definable multivariant functors on **Coh** is the denotation of an MLL + Mix proof. While dinaturality played a crucial role, another key fact was that **Coh** is fully and faithfully embedded into **GRel**. A somewhat related full completeness result for MLL using connections between **Coh** and Chu spaces was shown by Devarajan, Hughes, Plotkin, and Pratt [13]. This employs the stronger notion of relational parametricity [5], rather than dinaturality.

However it is impossible to extend Tan's full completeness theorem for **Coh** to Multiplicative Additive Linear Logic (MALL) because **Coh**, although it has (co)products, admits a variant of Berry's Gustave function which does not correspond to any proof. This was first mentioned by Girard [19] and is also a direct consequence of the Abramsky-Melliès' version

[3] of a 3-ary Gustave function in **GRel**. The history of this is discussed in [4] and also in Proposition 2.12.

One of the main advantages of Ehrhard's hypercoherences over coherence spaces is that they eliminate such functions. So there arises a natural question as to whether the dinatural interpretation of **HCoh** could provide a MALL fully complete model. The purpose of this paper is to provide an affirmative answer to this question. We prove that the dinatural interpretation over the double gluing category **GHCoh** is fully complete for MALL (*without* Mix) We also show in Section 7 that **HCoh** itself (without double gluing) does not permit a MALL+Mix full completeness theorem. Using double gluing on **HCoh** also allows us to eliminate the Mix rule. In fact, the status of this rule in the presence of additive connectives turns out to be a subtle problem (see Section 8).

One important notion we shall focus on is Joyal's *softness property* [30, 28]. Softness refers to a factorization property of morphisms between products and coproducts (see Section 2). In the case of lattices, it corresponds to an n-ary version of Whitman's property of free lattices [35, 30]. Moreover, by cut-elimination, the syntax of MALL (considered as a free category) satisfies softness; so this condition is necessary for any fully complete model.

1.2 Outline of the proof of MALL Full Completeness

Now let us outline the main ideas of our proof. We assume the framework of functorial polymorphism (see [9, 10, 24, 34, 33, 2]) which is an appropriate setting for our full completeness theorems. The theorem has three main steps:

(i) *Softness of* **HCoh** .

(ii) *Softness implies that Dinats yield* MALL *proof structures.*

(iii) *The Dinats in (ii) actually yield* MALL *proof nets.*

For (i), we begin by demonstrating the softness of Ehrhard's hypercoherences **HCoh** in Proposition 2.10 of Section 2. This will be shown by observing that there exists a sequence of intermediate $*$-autonomous categories $\{\mathbf{Coh_n} \mid 3 \leq \mathbf{n} \leq \omega\}$, where $\mathbf{Coh} = \mathbf{Coh_3}$ and $\mathbf{HCoh} = \mathbf{Coh}_\omega$. We show that $\mathbf{Coh_n}$ is m-ary soft for all $m < n$ but is not n-ary soft (Proposition 2.12 of Section 2); in particular, **HCoh** is soft.

In (ii), we develop an important consequence of softness (see Proposition 4.16 of Section 4). Let $\mathcal{C}$ be a $*$-autonomous category with (co)products. Suppose the dinats on $\mathcal{C}$ satisfy a softness condition, and are MLL+Mix fully complete and furthermore suppose that the `Mix` map is monic. Then every dinatural transformation ρ corresponds to a Girard MALL proof-structure Θ_ρ. The proof of this theorem proceeds via a preliminary full completeness theorem for certain fragments of MALL (see Theorem 4.1). In particular the theorem applies to **HCoh** (see Corollary 4.2). Hence we show that every dinatural transformation of **HCoh** corresponds to a MALL proof-structure Θ_ρ (see Corollary 4.55).

Recall that Girard introduced MALL proof structures as a natural extension to the additives of Danos-Regnier's MLL structures (see [14, 17]). These are obtained by enriching links and formulas with elements of certain boolean algebras while imposing some additional technical algebraic conditions. We interpret the above results as establishing one direction of the connection between Girard's MALL structures and Joyal's softness condition. More generally, an "equivalence" between MALL proof-structures and softness is discussed in the second author's paper [25].

In (iii), to show that the proof structures obtained above are actually MALL nets, we use the Loader-Hyland-Tan double-gluing construction, applied to the $*$-autonomous category **HCoh** . We obtain a category we call **GHCoh**, which does not satisfy the Mix rule. Our goal is to prove MALL full completeness for dinats on **GHCoh** (Theorem 6.4 of Section 6). A key observation is that there is a canonical inclusion *Dinat*-**GHCoh** $\hookrightarrow$ *Dinat*-**HCoh**, so we may use the previous results to guarantee **GHCoh** dinats also yield MALL proof-structures.

Let $PS(\rho)$ denote the set of proof structures associated to the dinat ρ by (ii). This set $PS(\rho)$ is nonempty. We assume for contradiction that ρ is not a denotation of any MALL proof. Then our association guarantees that proof-structures in $PS(\rho)$ enjoy certain important properties:

- the *unique link property*,
- the *no duplicate axiom link property* and
- contain certain simple oriented cycles (see below).

We will then show that this will lead us to a contradiction (to the fact that ρ is a dinatural family). Namely, using the embeddings **HCoh** $\hookrightarrow$ **Coh** $\hookrightarrow$ **GRel** , we construct an object of **Coh** , whose cliques and co-cliques intersect with cardinality ≥ 2. This is a contradiction.

We note that $PS(\rho)$ above is no longer necessarily a singleton, in sharp contrast to previous MLL full completeness proofs [1, 9]. In those papers, given a (di)natural transformation, one constructs a *unique* associated proof structure and then demonstrates that it must be a proof net. The contrast arises because, in our proof, we crucially use Girard's natural extension of the Danos/Regnier criterion for his MALL proof-structures. In this case, although Girard's criterion is simple enough, the possibility arises that from a single dinat there may be several different associated proof-structures. [1] Careful analysis is required to show that the associated set $PS(\rho)$ is sound (cf. Corollary 4.53); i.e., ρ is a denotation if and only if $\forall\,\Theta \in PS(\rho)$ Θ is a proof-net. Hence in (iii) we prove every element of $PS(\rho)$ is a MALL net, thus ρ is the denotation of a MALL proof (Theorem 6.2 of Section 5). Our proof uses a new characterization of cycles in such structures, which we call *simple oriented cycles*. Oriented cycles were first introduced by Abramsky-Melliès; but for the purposes of our proof, it suffices to cut down to a smaller class of what we call *simple cycles*, which we study in detail.

The paper is organized as follows: In Section 2 we introduce categories $\mathbf{Coh_n}$ intermediate between Girard's **Coh** and Ehrhard's **HCoh** and observe that **HCoh** is n-ary soft, for all natural numbers n. In Section 3 we show that *Dinat*-**HCoh** is fully complete for MLL + Mix . In Section 4 we prove that every dinatural transformation of **HCoh** corresponds to some Girard MALL proof-structure. In Section 5 we investigate simple oriented cycles in MALL proof-structures. In Section 6 we prove that the proof-structure associated to every dinatural transformation of **GHCoh** is a proof-net for MALL; i.e., we obtain MALL full

[1]Recently Hughes and van Glabbeek [27] have considerably extended our understanding of the theory of additive proof structures. This is discussed in the conclusion.

completeness in *Dinat*-**GHCoh** . In Section 7 we discuss the Mix rule in the presence of the additives.

1.3 Related Works

The first dinatural full completeness theorem for MALL was established in the work of Abramsky-Melliès [2, 3]. This work extended the game-theoretic full completeness theorems for MLL+Mix by Abramsky and Jagadeesan [1] by introducing the notion of *concurrent games*. In this setting, certain winning strategies yield dinatural transformations which denote MALL proofs. Both the results of Abramsky-Melliès and our own work can be considered as enriching **Rel**-models with additional structure.

The preliminary stages of the present paper were influenced by considering what we here call *the Abramsky-Melliès Gustave function*. Abramsky and Melliès also gave a detailed study of certain oriented cycles in MALL proof structures. As previously mentioned, these ideas also influenced the work here; however our presentation is self contained and uses the more restricted notion of simple cycle.

In a quite different direction, Girard's recent work on *ludics* and *the logic of rules* [21, 20] establishes a full completeness theorem for MALL , although not using the dinatural framework. Ludics is a drastic reinterpretation of the semantics of proof theory, combining ideas from proof search and cut-elimination into a kind of abstract game semantics. It would be very interesting to obtain explicit connections between ludics and our hypercoherence-based fully-complete models.

It would be important to find a relationship between our complicated association of proof-structures with dinats and Hughes-Glabbeek [27]'s new notion of MALL proof-structures and their associated correctness criterion which is stronger than Girard's original notion. We make further comments on their work in the conclusion.

Notation 1.1 Let A denote a set and $P(A)$ the power set of A . We denote the finite power set $P_{fin}(A) := \{\alpha \in P(A) \mid \alpha \text{ is a finite set}\}$. $P^*_{fin}(A) := P_{fin}(A) \setminus \{\emptyset\}$. $P^{(*)}_{<n}(A) := \{\alpha \in P^{(*)}_{fin}(A) \mid \#\alpha < n\}$, where $\#$ denotes the cardinality. We write $X \subseteq^*_{fin} Y$ when X is a finite and non-empty subset of Y and write $X \subseteq^*_{<n} Y$ when X is a non-empty subset of Y such that $\#X < n$. $A \times B$ denotes the cartesian product of sets A and B. For $C \subseteq A \times B$, we use $\pi_1(C) := \{a \in A \mid \exists b \in B\ (a,b) \in C\}$ for its first projection and use $\pi_2(C) := \{b \in B \mid \exists a \in A\ (a,b) \in C\}$ for its second projection. $A + B$ denotes the disjoint union of sets A and B, i.e., $A + B := \{(1,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}$. For $C \subseteq A + B$, we use $C_1 := \{a \in A \mid (1,a) \in D\}$ for its first component and use $C_2 := \{b \in B \mid (2,b) \in D\}$ for its second component.

We denote vectors of quantities by underlining or overlining (depending on ease of reading), so for example $\underline{A}$ or $\overline{A}$ denote vectors $(A_1, \cdots, A_n)$ of length n, for some n. Multivariant functors $F : (\mathcal{C}^{op})^n \times \mathcal{C}^n \to \mathcal{C}$ are denoted on objects by $F(\underline{X}; \underline{Y})$, for $\underline{X}, \underline{Y} \in \mathcal{C}^n$.

2 Categories $\mathbf{Coh_n}$ of n-coherences and m-ary softness of $\mathbf{Coh_n}$, $m < n \leq \omega$

2.1 Categories of n-coherences

The purpose of this section is to introduce the categories $\mathbf{Coh_n}$ for $2 < n \leq \omega$, which are intermediate between Girard's **Coh** [17], which is $\mathbf{Coh_3}$, and Ehrhard's **HCoh** [15], which is $\mathbf{Coh}_\omega$ in our terminology. Consequently there arises a hierarchy of coherent spaces $\mathbf{Coh_n}$ between **Coh** and **HCoh**. The existence of such a hierarchy is part of the folklore; e.g., Lamarche [31] also discussed it under the name of Girard quantale-valued sets. However one of our contributions in this section is to establish that the categories of these hierarchical coherent spaces are soft (Proposition 2.12). In particular our result on softness of **HCoh** (Corollary 2.11) is exactly a counterpart of Ehrhard's first order sequentiality, which is the origin of his discovery of **HCoh**. For hypercoherences, we often follow the text of Amadio-Curien [4] in addition to Ehrhard [15].

Definition 2.1 (n-coherence E) An *n-coherence E* is a pair

$$E := (|E|, \Gamma(E))$$

where $|E|$ is a set and $\Gamma(E) \subseteq P^*_{<n}(|E|)$ such that $\forall a \in |E| \quad \{a\} \in \Gamma(E)$.

We use the notation $\Gamma^*(E) := \{u \in \Gamma(E) \mid \#u > 1\}$. An n-coherence E is identified with a hypergraph, each of whose edges is a set of vertices of cardinality less than n: namely $|E|$ determines the set of nodes and each element of $\Gamma(E)$ determines a hyperedge on $|E|$.

Definition 2.2 (the set $D(E)$ of states for an n-coherence E) For an n-coherence E, the set $D(E)$ of *states* for E is

$$D(E) := \{X \subseteq |E| \mid \forall u \subseteq^*_{<n} X \; u \in \Gamma(E)\}$$

where $B \subseteq^*_{<n} A$ means B is a nonempty subset of A of cardinality $< n$.

Definition 2.3 (linear implication of n-coherences) For n-coherences E and F, the n-coherence $E \multimap F$, called *linear implication* of E and F, is

$$E \multimap F := (|E| \times |F|, \Gamma(E \multimap F))$$

where $w \in \Gamma(E \multimap F)$ iff

(i) $w \subseteq |E| \times |F|,\ \#w < n$

(ii) $\pi_1(w) \in \Gamma(E) \Rightarrow (\pi_2(w) \in \Gamma(F) \land \ (\#\pi_2(w) = 1 \Rightarrow \#\pi_1(w) = 1))$

Definition 2.4 (the intermediate category $\mathbf{Coh_n}$) The category $\mathbf{Coh_n}$ consists of the following: objects: n-coherences $E := (|E|, \Gamma(E))$
morphisms: $\mathbf{Coh_n}(E, F) := D(E \multimap F)$

Remark 2.5 A morphism is a relation on hypergraphs which "maps" hyperedges to hyperedges and such that the preimage of a loop is a loop (but in general the preimage of a hyperedge is not necessarily a hyperedge).

It can be checked (as in Proposition 5 of [15]) that the above data indeed defines a category: For $E, F \in \mathbf{Coh}_n$

1. $Id_E := \{(a,a) \mid a \in |E|\} \in D(E \multimap E)$
2. If $R \in D(E \multimap F)$ and $S \in D(F \multimap G)$ then

$$S \circ R := \{(a,c) \mid \exists b((a,b) \in R \wedge \ (b,c) \in S)\} \in D(E \multimap G).$$

Proposition 2.6 $\mathbf{Coh_n}$ *becomes a $*$-autonomous category with products and coproducts.*

We indicate the structure on objects, following [4]:

(linear negation:) $E^\perp := (|E|, \Gamma(E^\perp))$ where

$$\Gamma^*(E^\perp) := P^*_{<n}(|E|) \setminus \Gamma^*(E).$$

(tensor:) $E \otimes F := (|E| \times |F|, \Gamma(E \otimes F))$ where

$$w \in \Gamma(E \otimes F) \text{ iff } \begin{array}{l} w \subseteq |E| \times |F|, \#w < n \text{ and} \\ (w_1 \in \Gamma(E) \wedge \ w_2 \in \Gamma(F)). \end{array}$$

(product:) $E \& F = (|E| + |F|, \Gamma(E \& F))$ where

$$w \in \Gamma(E \& F) \text{ iff } \begin{array}{l} w \subseteq |E| + |F|, \#w < n \text{ and} \\ (w_2 = \emptyset \Rightarrow w_1 \in \Gamma(E)) \wedge \ (w_1 = \emptyset \Rightarrow w_2 \in \Gamma(F)). \end{array}$$

Hence we have by de Morgan duality:

(par:) $E \parr F := (|E| \times |F|, \Gamma(E \parr F))$ where

$$w \in \Gamma^*(E \parr F) \text{ iff } \begin{array}{l} w \subseteq |E| \times |F|, \#w < n \text{ and} \\ (w_1 \in \Gamma^*(E) \vee w_2 \in \Gamma^*(F)). \end{array}$$

(coproduct:) $E \oplus F := (|E| + |F|, \Gamma(E \oplus F))$ where

$$w \in \Gamma(E \oplus F) \text{ iff } \begin{array}{l} w \subseteq |E| + |F|, \#w < n \text{ and} \\ (w_1 \in \Gamma(E) \wedge \ w_2 = \emptyset) \vee (w_1 = \emptyset \wedge \ w_2 \in \Gamma(F)). \end{array}$$

$\mathbf{1}$ denotes the unique n-coherence such that $|\mathbf{1}|$ is the singleton $\{\star\}$. Then $\mathbf{1} = \mathbf{1}^\perp$ and $\mathbf{1}$ becomes the unit both for $\otimes$ and $\parr$.

$\mathbf{Coh_n}$ has additional canonical morphisms $\texttt{Mix} : E \otimes F \to E \parr F$, which are given by $Id_{|E| \times |F|}$. Note that $\texttt{Mix}$ is monic in $\mathbf{Coh_n}$.

Remark 2.7

(i) It appears that the definition of coproduct is somehow more "natural". So we could equally take the coproduct as primitive and define the product by de Morgan duality.

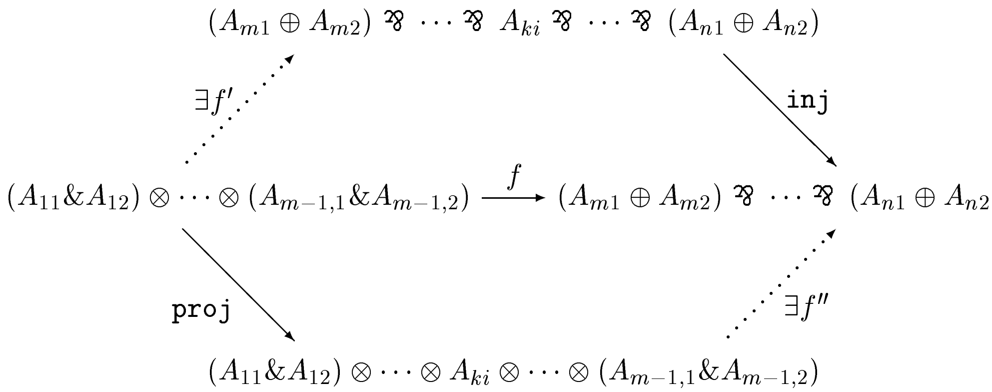
(ii) Observe that $\mathbf{Coh_2}$ is exactly the category **Rel**, whose objects are sets, whose morphisms are binary relations, and where composition means relational composition.

(iii) The above definition of n-coherence is an intermediate notion to Girard's coherences and Ehrhard's hypercoherences, in that if $n = 3$ we obtain the category **Coh** and if $n = \omega$ we obtain the category **HCoh**.

2.2 n-ary softness and double gluing

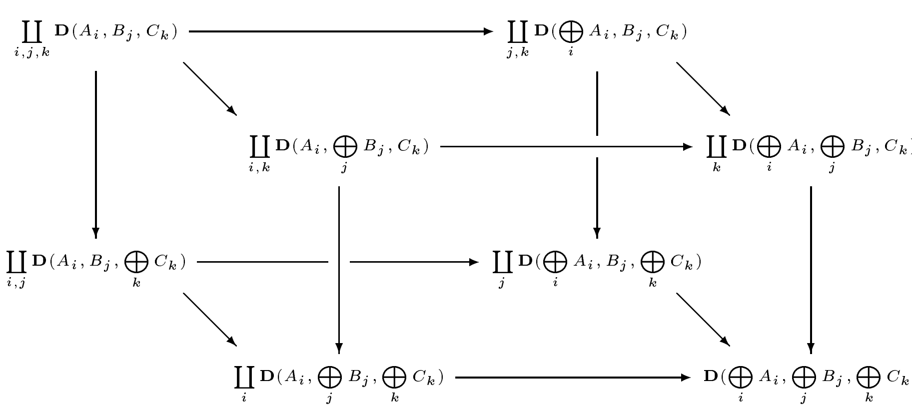
Before going to Proposition 2.10, we remind the reader of the definition of n-ary softness due to Joyal [30].

Definition 2.8 (n-ary softness (cf. Joyal [30])) A morphism f is called *n-ary soft* when the following holds: if f is of the form $(A_{11}\&A_{12})\otimes\cdots\otimes(A_{m-1,1}\&A_{m-1,2})\longrightarrow(A_{m1}\oplus A_{m2}) \parr \cdots \parr (A_{n1}\oplus A_{n2})$, then there exists k $(1\leq k\leq n)$ such that f factors through either a product projection from $A_{k1}\&A_{k2}$ $(k<m)$ or a coproduct injection into $A_{k1}\oplus A_{k2}$ $(k\geq m)$; namely, either of the following two triangle diagrams commutes:



A $*$-autonomous category with products and coproducts is called *soft* if all its morphisms are soft. In a $*$-autonomous category with products and coproducts, the above f is transposed into $\hat{f}:\mathbf{1}\rightarrow(A_{11}^{\perp}\oplus A_{12}^{\perp}) \parr \cdots \parr (A_{m-1,1}^{\perp}\oplus A_{m-1,2}^{\perp}) \parr (A_{m1}\oplus A_{m2}) \parr \cdots \parr (A_{n1}\oplus A_{n2})$ and vice versa. Hence it suffices to consider the case with $m=1$, in which case the lower triangle in the diagram does not exist.

Observe that for a $*$-autonomous category $\mathcal{C}$ with products and coproducts, the condition that all the dinats of $\mathcal{C}$ are n-ary soft can be characterized by means of an n-dimensional weak pushout (cf. Joyal [30]). E.g., when $n=3$ the condition is equivalent to the fact that the following cube is a 3-dimensional weak pushout, where $\mathbf{D}$ denotes the functor $\mathcal{C}\times\mathcal{C}\times\mathcal{C}\rightarrow\mathbf{Set}$ defined by $\mathbf{D}(A,B,C):=\mathcal{C}(\mathbf{1},A \parr B \parr C)$ and $\coprod$ denotes disjoint union in $\mathbf{Set}$.



We observe that originally Joyal required the above diagram to be a pushout, not just a weak pushout. The weak notion suffices for our purposes here, and corresponds closer to the syntax, as in the following remark.

Remark 2.9 (Necessity of Softness) Softness is a necessary condition for a MALL full completeness theorem. First, observe that the syntax is "soft" in the following sense: if we consider the representation of a cut-free proof of a sequent representing a morphism, say $(A_{11}\& A_{12}) \otimes \cdots \otimes (A_{m-1,1}\& A_{m-1,2}) \vdash (A_{m1} \oplus A_{m2}) \parr \cdots \parr (A_{n1} \oplus A_{n2})$ it must end with either a &-left, or a $\oplus$-right rule[2]. This guarantees softness for any fully complete categorical model as follows: by abuse of notation, if in a model we have a morphism $(A_{11}\& A_{12}) \otimes \cdots \otimes (A_{m-1,1}\& A_{m-1,2}) \to (A_{m1} \oplus A_{m2}) \parr \cdots \parr (A_{n1} \oplus A_{n2})$, by fullness this arises from a (cut-free) proof of a sequent as above. Hence by the softness of the syntax, the proof factors through either a projection on the left or an injection on the right. By the Soundness Theorem, this factorization is transformed (by the interpretation of the syntax in the model) into a factorization of the original morphism.

Proposition 2.10 $\mathbf{Coh_n}$ *is m-ary soft for all $m < n \leq \omega$.*

Proof. We assume $m \geq 2$ since the assertion when $m = 1$ automatically holds by virtue of the definition of product. Suppose for deriving a contradiction that there exists a morphism $R : (E_{1,1}\& E_{1,2}) \otimes \cdots \otimes (E_{m-1,1}\& E_{m-1,2}) \longrightarrow (E_{m,1} \oplus E_{m,2})$ in $\mathbf{Coh_n}$ such that R does not factor through any & explicitly appearing in the domain nor through the $\oplus$ explicitly appearing in the codomain. Note for example, to say that R factors through $E_{1,1}\& E_{1,2}$ means that there exists a $j \in \{1,2\}$ such that for all vectors $\mathbf{x} = (x_1, \cdots, x_m) \in R \subseteq |E_{1,1}\& E_{1,2}| \times \cdots \times |E_{m-1,1}\& E_{m-1,2}| \times |E_{m,1} \oplus E_{m,2}|$, it follows that $\{x_1\}_j = \emptyset$, i.e. $x_1 \notin |E_{1,j}|$.

We shall choose a subset $u \subseteq R$, whose cardinality is m,

$$u := \{\mathbf{x}^i := (x_1^i, x_2^i, \ldots, x_m^i)\}_{1 \leq i \leq m}$$

where $x_k^i \in |E_{k,1}\& E_{k,2}|, 1 \leq k \leq m-1$ and $x_m^i \in |E_{m,1} \oplus E_{m,2}|$ by induction on i as follows:

(For $\mathbf{x}^1$ and $\mathbf{x}^2$) By our supposition, R does not factor through the first or the second component. So $\mathbf{x}^1 := (x_1^1, x_2^1, \ldots)$ and $\mathbf{x}^2 := (x_1^2, x_2^2, \ldots)$ can be chosen such that $\forall j \in \{1,2\}(\{x_1^1, x_1^2\}_j \neq \emptyset \wedge \ \{x_2^1, x_2^2\}_j \neq \emptyset)$.

(For $\mathbf{x}^{i+1}$) The $i+1$-st component $x_{i+1}^{i+1} \in | E_{i+1,1}\& E_{i+1,2} |$ of $\mathbf{x}^{i+1}$ can be chosen as follows: By considering the set $v := \{x_{i+1}^1, x_{i+1}^2, \ldots, x_{i+1}^{i+1}\}$ of the $i+1$-st components for $\mathbf{x}^k$ with $1 \leq k \leq i$, we can take x_{i+1}^{i+1} such that $\forall j \in \{1,2\}$ $(v \cup \{x_{i+1}^{i+1}\})_j \neq \emptyset$ by virtue of the fact that R does not factor through the $i+1$-st component.

For such a choice of subset u of cardinality m, we have

$$\forall i \in \{1, \ldots, m\}\ \forall j \in \{1,2\} \quad (\pi_i(u))_j \neq \emptyset \tag{1}$$

This condition implies that if we project to the first $m-1$ components, we obtain $\pi_{1,\ldots,m-1}(u) \in \Gamma((E_{1,1}\& E_{1,2}) \otimes \cdots \otimes (E_{m-1,1}\& E_{m-1,2}))$. Thus $\pi_m(u) \in \Gamma(E_{m,1} \oplus E_{m,2})$ since $u \in \Gamma((E_{1,1}\& E_{1,2}) \otimes \cdots \otimes (E_{m-1,1}\& E_{m-1,2}) \multimap (E_{m,1} \oplus E_{m,2}))$. Hence $\exists j \in \{1,2\}(\pi_m(u))_j = \emptyset$ from the definition of $\oplus$. This is a contradiction to (1) when $i = m$. □

Corollary 2.11 (softness of HCoh) **HCoh** *is n-ary soft for all natural numbers n.*

Proof. This follows because **HCoh** is $\mathbf{Coh}_\omega$. □

[2]Strictly speaking, proof theorists would replace the $\otimes$'s on the left side and $\parr$'s on the right side of the sequent by commas.

Proposition 2.12 (Existence of n-ary Gustave functions) *If $2 < n < \omega$ then* $\mathbf{Coh_n}$ *is not n-ary soft.*

Proof. For objects $E_1, \ldots, E_{n-1}$, let D denote the following object in $\mathbf{Coh_n}$:

$$((E_1 \& \cdots \& E_{n-1}) \oplus E_n) ⅋ ((E_2 \& \cdots \& E_n) \oplus E_1) ⅋ \cdots$$

$$\cdots ⅋ ((E_{n-1} \& \cdots \& E_{n-3}) \oplus E_{n-2}) ⅋ ((E_n \& \cdots \& E_{n-2}) \oplus E_{n-1})$$

$$\text{where } E_n := E_1^{\perp} \otimes \cdots \otimes E_{n-1}^{\perp}.$$

Observe that this object denotes a provable formula of MALL. Let S_n be the symmetric group on n. For $\sigma \in S_n$, $R_\sigma \subseteq |D|$ is defined by

$$R_\sigma := \left\{ ((\sigma(1), a_1), \ldots, (\sigma(k), a_k), \ldots, (\sigma(n), a_n)) \;\middle|\; \begin{array}{ll} a_k \in |E_{\sigma(k)}| & \text{if } \sigma(k) \neq n \\ a_k := (a_1, \ldots, a_{n-1}) & \text{if } \sigma(k) = n \end{array} \right\}$$

In particular when σ is the cyclic permutation $(n, n-1, \ldots, 2, 1)$, R_σ is exactly the denotation of a proof of the formula denoted above, hence $R_\sigma \in \mathbf{Coh_n}(\mathbf{1}, D)$.

Now we define $R \in \mathbf{Coh_n}(\mathbf{1}, D)$ to be the union of R_σ when σ runs over $S'_n := S_n \setminus \{(n, n-1, \ldots, 2, 1)\}$:

$$R := \bigcup_{\sigma \in S'_n} R_\sigma$$

R does not factor through any explicitly occurring $\oplus$, i.e. R is not n-ary soft. The morphism R is called an *n-ary Gustave function.* □

For example, when $n = 3$ in the above proof, we obtain the following Gustave function, first discussed by Girard [19] and also by Abramsky and Melliès [3]:

$$\begin{array}{rl} R \;:= & \{((1,a_1),(2,a_3),(3,a_2)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ \cup & \\ & \{((3,a_3),(1,a_2),(2,a_1)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ \cup & \\ & \{((2,a_2),(3,a_1),(1,a_3)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ \cup & \\ & \{((1,a_1),(1,a_2),(1,a_3)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ \cup & \\ & \{((2,a_2),(2,a_3),(2,a_1)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \end{array}$$

It was shown by Tan [34] that **Coh** (in our terminology $\mathbf{Coh_3}$) is fully and faithfully embedded into Loader's category **GRel** of *linear logical predicates* [33]. This construction has been generalized by Hyland and Tan [34] to a general *double gluing* construction $\mathbf{G}\mathcal{C}$ over certain categories $\mathcal{C}$. This is described later in Section 3.2 .

Definition 2.13 (GRel (cf. Loader [33] and Tan [34])) **GRel** denotes *the double gluing category over the category* **Rel** defined as follows:

Objects: triples $\mathcal{A} = (|\mathcal{A}|, \mathcal{A}_p, \mathcal{A}_{cp})$ where $|\mathcal{A}|$ is an object of **Rel**, $\mathcal{A}_p \subseteq \mathbf{Rel}(I, |\mathcal{A}|)$ and $\mathcal{A}_{cp} \subseteq \mathbf{Rel}(|\mathcal{A}|, I)$.

Morphisms: A morphism $f : \mathcal{A} \to \mathcal{B}$ of **GRel** is a morphism $R : |\mathcal{A}| \to |B|$ of **Rel** such that the following conditions hold:

$$\begin{array}{ll}\text{(image condition:)} & \forall \alpha \in \mathcal{A}_p \ [\alpha]R := \{b \in \beta \mid \exists\, a \in \alpha (a, b) \in R\} \in \mathcal{B}_p \\ \text{(co-image condition:)} & \forall \beta \in \mathcal{B}_{cp} \ R[\beta] := \{a \in \alpha \mid \exists\, b \in \beta (a, b) \in R\} \in \mathcal{A}_{cp}\end{array}$$

GRel becomes a $*$-autonomous category with products and coproducts, given by the following structure on objects:

(linear negation:) $\mathcal{A}^\perp = (|\mathcal{A}|, \mathcal{A}_{cp}, \mathcal{A}_p)$.
(tensor:) the tensor $\mathcal{A} \otimes \mathcal{B}$ is defined by $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and

$$\begin{array}{l}(\mathcal{A} \otimes \mathcal{B})_p = \{\alpha \times \beta \mid \alpha \in \mathcal{A}_p \text{ and } \beta \in \mathcal{B}_p\} = \mathcal{A}_p \times \mathcal{B}_p \\ (\mathcal{A} \otimes \mathcal{B})_{cp} = \mathbf{GRel}(\mathcal{A}, \mathcal{B}^\perp) = \mathbf{GRel}(\mathcal{B}, \mathcal{A}^\perp)\end{array}$$

$\mathbf{1} := (I, \{id_I\}, \mathbf{Rel}(I, I))$ becomes the tensor unit.

(product:) the product $\mathcal{A} \& \mathcal{B}$ is defined by $|\mathcal{A} \& \mathcal{B}| := |\mathcal{A}| + |\mathcal{B}|$ and

$$\begin{array}{l}(\mathcal{A} \& \mathcal{B})_p := \{\alpha + \beta \mid \alpha \in \mathcal{A}_p \text{ and } \beta \in \mathcal{B}_p\} := \mathcal{A}_p \frown \mathcal{B}_p \\ (\mathcal{A} \& \mathcal{B})_{cp} = \mathcal{A}_{cp} + \mathcal{B}_{cp}\end{array}$$

Hence we have by de Morgan duality:

(par:) $\mathcal{A} \parr \mathcal{B} := (\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$: Explicitly

$$\begin{array}{l}(\mathcal{A} \parr \mathcal{B})_p = \mathbf{GRel}(\mathcal{A}^\perp, \mathcal{B}) = \mathbf{GRel}(\mathcal{B}^\perp, \mathcal{A}) \\ (\mathcal{A} \parr \mathcal{B})_{cp} = \{\alpha' \times \beta' \mid \alpha' \in \mathcal{A}_{cp} \text{ and } \beta' \in \mathcal{B}_{cp}\} := \mathcal{A}_{cp} \times \mathcal{B}_{cp}\end{array}$$

(coproduct:) $\mathcal{A} \oplus \mathcal{B} := (\mathcal{A}^\perp \& \mathcal{B}^\perp)^\perp$: Explicitly

$$\begin{array}{l}(\mathcal{A} \oplus \mathcal{B})_p = \mathcal{A}_p + \mathcal{B}_p \\ (\mathcal{A} \oplus \mathcal{B})_{cp} = \{\alpha + \beta \mid \alpha \in \mathcal{A}_{cp} \text{ and } \beta \in \mathcal{B}_{cp}\} = \mathcal{A}_{cp} \frown \mathcal{B}_{cp}\end{array}$$

Recall from Remark 2.7 that $\mathbf{Coh}_3$ is Girard's category **Coh** of coherence spaces.

Proposition 2.14 (Tan [34]) [3] **Coh** *is equivalent to the full subcategory of* **GRel** *consisting of the objects* $\mathcal{A} := (|\mathcal{A}|, \mathcal{A}_p, \mathcal{A}_{cp})$ *satisfying:*

- $\alpha \in \mathcal{A}_p$ *iff* $\forall \beta \in \mathcal{A}_{cp} \ \#(\alpha \cap \beta) \leq 1$
- $\beta \in \mathcal{A}_{cp}$ *iff* $\forall \alpha \in \mathcal{A}_p \ \#(\alpha \cap \beta) \leq 1$
- $|\mathcal{A}| = \bigcup_{\alpha \in \mathcal{A}_p} \alpha = \bigcup_{\beta \in \mathcal{A}_{cp}} \beta$

[3] It may be shown that $\mathbf{Coh_{n+1}}$ is a full subcategory of the iterated double gluing category $\mathbf{G^{n-1}Rel}$. This is studied further in a tech report [26].

3 Multiplicative full completeness of HCoh and GHCoh

3.1 MLL+Mix full completeness of $\mathbf{Coh_n}$ with $2 < n \leq \omega$

We assume familiarity with dinatural transformations, hereafter *dinats*, and functorial polymorphism, see [5, 7, 9, 22]). This is the most appropriate setting for our full completeness theorems.

Definition 3.1 *Dinat-*$\mathcal{C}$ denotes the structure whose objects are MALL-definable multivariant functors in $\mathcal{C}$ and whose morphisms are dinatural transformations between them

From now on, dinatural transformations will always be assumed to be between definable functors in some (perhaps proper) fragment of MALL. As is well-known, *Dinat-*$\mathcal{C}$ is not in general a category, since dinaturals need not compose. One of the interesting consequences of a full completeness theorem (for a fragment of linear logic) is that dinaturals do form a category, but we do not know this fact until *after* we have proven the theorem! The reason is that syntax is compositional and a fully complete modelling has a precise correspondence to the syntax. Nonetheless, we will use categorical terminology when referring to the structure *Dinat-*$\mathcal{C}$, as if it were a category.

In the Introduction, we discussed the problem of *full completeness* for a logic $\mathcal{L}$ with respect to a categorical structure $\mathcal{M}$. In what follows, the categorical structure $\mathcal{M}$ will be *Dinat-*$\mathcal{C}$. In this structure, we interpret (one-sided) sequents $\vdash \Gamma$ as multivariant functors, as usual in functorial polymorphism [5, 9]. We interpret proofs of sequents $\vdash \ \Gamma$ as dinatural transformations of the form $\mathbf{1} \to [\![\Gamma]\!]$, where $\mathbf{1}$ is the constant functor with value the tensor unit $\mathbf{1}$. Full completeness now becomes the statement: *Every dinat* $\mathbf{1} \to [\![\Gamma]\!]$ *is the denotation of a proof.* The MLL full completeness theorems in this section are all fully-faithful representations.

Tan [34] proved the following multiplicative full completeness theorem via the full and faithful embedding : $\mathbf{Coh_3} \hookrightarrow \mathbf{GRel}$, where Mix is the inference rule:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ \mathsf{Mix}$$

Fact 3.2 (Tan [34]) *Dinat-*$\mathbf{Coh_3}$ *is fully complete for* MLL + Mix *.*

For an object $E \in \mathbf{Coh_n}$ and $m < n$, we can define $\Gamma_{<m}(E) := \{X \in \Gamma(E) \mid \#X < m\}$. Then $(|E|, \Gamma_{<m}(E))$ is an object of $\mathbf{Coh_m}$.

Definition 3.3 (functor U_{nm}) *Let* $3 \leq m < n \leq \omega$*. Then the functor*

$$U_{nm} : \mathbf{Coh_n} \to \mathbf{Coh_m}$$

is defined by mapping $(|E|, \Gamma(E))$ *to* $(|E|, \Gamma_{<m}(E))$ *and* $R : E \multimap F$ *to* $R : U_{nm}(E) \multimap U_{nm}(F)$. U_{nm} *is full and preserves* $*$*-autonomy, as well as (co)products. Composition of functors satisfies* $U_{ml} \circ U_{nm} = U_{nl}$.

Remark 3.4 As in Ehrhard [15], we can define the functor $\mathbf{PN} : \mathbf{Coh_n} \to (\mathbf{Coh_n}^-)^+$, where $\subseteq^*_{fin}$ in his definitions of positive/negative hypercoherences is replaced by $\subseteq^*_{<n}$. Then U_{n3} can be identified with $\mathbf{PN}$ because $(\mathbf{Coh_n}^-)^+$ can be considered as $\mathbf{Coh}$.

Lemma 3.5 *Let* $\rho \in$ *Dinat-*$\mathbf{Coh_n}(A(\underline{X}; \underline{X}), B(\underline{X}; \underline{X}))$*. If* $\underline{E}$ *and* $\underline{F}$ *are vectors of objects from* $\mathbf{Coh_n}$ *such that* $U_{n3}(\underline{E}) = U_{n3}(\underline{F})$ *then* $U_{n3}(\rho_{\underline{E}}) = U_{n3}(\rho_{\underline{F}})$*.*

Proof. For each object $E \in \mathbf{Coh_n}$, consider $E^\bullet := (|E|, \Gamma_{<3}(E)) \in \mathbf{Coh_n}$. Then it can be checked that $Id_{|E|} \in \mathbf{Coh_n}(E^\bullet, E)$. Moreover $U_{n3}(\underline{E}) = U_{n3}(\underline{F})$ implies that $E^\bullet = F^\bullet$. Thus it suffices to prove that $U_{n3}(\rho_E) = U_{n3}(\rho_{E^\bullet})$. But this is obvious by chasing the hexagonal diagram of dinaturality for $Id_{|E|} : E^\bullet \to E$. □

Given a dinatural transformation $\rho := \{\rho_{E'} : A(E'; E') \to B(E'; E')\}$ in $\mathbf{Coh_n}$, let us apply the functor U_{n3}, say U for short, to every morphism $\rho_{E'}$. Then we have the **Coh**-morphism $U(\rho_{E'}) = \rho_{E'} : A(U(E'); U(E')) \to B(U(E'); U(E'))$ since U preserves $*$-autonomy with (co)products. By Lemma 3.5, if $U(E') = U(F')$ then $U(\rho_{E'}) = U(\rho_{F'})$. Thus $U(\rho)$ determines the following family, say $\mathcal{J}_n(\rho)$, of morphisms indexed by **Coh** objects:

$$\mathcal{J}_n(\rho) := \{\mathcal{J}_n(\rho)_E := \rho_{E'} : A(E; E) \to B(E; E) \mid E = U(E') \text{ and } E \in \mathbf{Coh}\}$$

The fact that U is full assures that $\mathcal{J}_n(\rho)$ becomes a dinatural transformation in **Coh**: the condition $B(id_E; R) \circ \mathcal{J}_n(\rho)_E \circ A(R; id_E) = B(R; id_F) \circ \mathcal{J}_n(\rho)_F \circ A(id_F; R)$ should be checked for every $R \in \mathbf{Coh}(E, F)$. Since U is full, $\exists\ E', F' \in \mathbf{Coh_n}$ such that $E = U(E')$, $F = U(F')$ and $R \in \mathbf{Coh_n}(E', F')$. Thus we have the condition in $\mathbf{Coh_n}$ that $B(id_{E'}; R) \circ \rho_{E'} \circ A(R; id_{E'}) = B(R; id_{F'}) \circ \rho_{F'} \circ A(id_{F'}; R)$, from which we can derive the required condition in **Coh** by applying the functor U. Moreover the functor $\mathcal{J}_n$ so defined satisfies the following:

Proposition 3.6 (The faithful functor $\mathcal{J}_n$) *The full functor U_{n3} determines a faithful functor*

$$\mathcal{J}_n : \textit{Dinat-}\mathbf{Coh_n} \longrightarrow \textit{Dinat-}\mathbf{Coh}.$$

Note that $\mathcal{J}_n$ is not full since *Dinat*-$\mathbf{Coh_n}$ with $n > 3$ is 3-ary soft but *Dinat*-**Coh** is not 3-ary soft. Note also that $\mathcal{J}_n$ preserves composition, when defined.

Proof. Take dinatural transformations ρ and σ of $\mathbf{Coh_n}$ such that $\mathcal{J}_n(\rho) = \mathcal{J}_n(\sigma)$. Analogously to Remark 3.4 above, and by Sections 5 and 6 of [15], we can define the inclusion functor $\mathbf{I}_n^+ : \mathbf{Coh_3} \to \mathbf{Coh_n}$ when $\subseteq^*_{fin}$ in Ehrard's definition of positive hypercoherences is replaced by $\subseteq^*_{<n}$. Now $\mathcal{J}_n(\rho) = \mathcal{J}_n(\sigma)$ is equivalent to say that if $\underline{E}$ is a vector of objects from the image of $\mathbf{I}_n^+$ then $\rho_{\underline{E}} = \sigma_{\underline{E}}$. Thus with the help of Lemma 3.5 ρ and σ are the same since for all $E \in \mathbf{Coh_n}$ there exists $E' \in \mathbf{I}_n^+(\mathbf{Coh_3})$ such that $U(E) = U(E')$. □

Fact 3.2 together with Proposition 3.6 implies the following:

Proposition 3.7 (MLL+Mix full completeness)
For $2 < n \leq \omega$, Dinat-$\mathbf{Coh_n}$ is fully complete for MLL + Mix .

The above multiplicative full completeness theorem for *Dinat*-$\mathbf{Coh_n}$ cannot be extended to the level of MALL+Mix if $n \neq \omega$ (and even for $n = \omega$ we must introduce double gluing to get MALL full completeness, as we show below). The reason for the failure is that the categories *Dinat*-$\mathbf{Coh_n}$, $n \neq \omega$ fail to be soft:

Proposition 3.8 *For all $n < \omega$, the categories Dinat-$\mathbf{Coh_n}$ are not n-ary soft and hence fail to be* MALL+Mix *fully complete.*

Proof. The n-ary Gustave functions in Proposition 2.12 can be shown to be the components of a dinatural transformation R. □

Hence from now on we shall restrict our attention to $\mathbf{Coh}_\omega = \mathbf{HCoh}$.

3.2 The Double Gluing Construction

We now present the Hyland-Tan double gluing construction. We will follow Chapter 1 of Tan [34], observing that the gluing construction applies to general $*$-autonomous categories, not just compact closed ones.

Definition 3.9 Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, (-)^\perp)$ be a $*$-autonomous category. Let H denote the covariant *points* functor $\mathcal{C}(\mathbf{1}, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ and K denote the contravariant *copoints* functor $\mathcal{C}(-, \mathbf{1}^\perp) \cong \mathcal{C}(\mathbf{1}, (-)^\perp) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$.

We define a new category, $\mathbf{G}\mathcal{C}$, the *double gluing category* of $\mathcal{C}$, whose objects are triples $\mathcal{A} = (A, \mathcal{A}_p, \mathcal{A}_{cp})$ where $A := |\mathcal{A}|$ is an object of $\mathcal{C}$, where $\mathcal{A}_p \subseteq H(|\mathcal{A}|) = \mathcal{C}(\mathbf{1}, A)$ is a set of points of A and $\mathcal{A}_{cp} \subseteq K(|\mathcal{A}|) = \mathcal{C}(A, \mathbf{1}^\perp) \cong \mathcal{C}(\mathbf{1}, A^\perp)$ is a set of copoints of A .

A morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{G}\mathcal{C}$ is a morphism $f : |\mathcal{A}| \longrightarrow |\mathcal{B}|$ in $\mathcal{C}$ such that $Hf : \mathcal{A}_p \longrightarrow \mathcal{B}_p$ and $Kf : \mathcal{B}_{cp} \longrightarrow \mathcal{A}_{cp}$ are well-defined $\mathbf{Set}$-maps, i.e. $f(\mathcal{A}_p) \subseteq \mathcal{B}_p$ and $f^\perp(\mathcal{B}_{cp}) \subseteq \mathcal{A}_{cp}$.

Given $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{C}$ in $\mathbf{G}\mathcal{C}$, the composition $gf : \mathcal{A} \longrightarrow \mathcal{C}$ is induced from the underlying composition in $\mathcal{C}$. Similarly, the identity morphism on $\mathcal{A}$ is given by the identity morphism on $|\mathcal{A}|$ in $\mathcal{C}$.

Fact 3.10 *For any $*$-autonomous category $\mathcal{C}$, $\mathbf{G}\mathcal{C}$ is a $*$-autonomous category.*

Proof. We first describe the tensor product $\mathcal{A} \otimes \mathcal{B}$:

$$\mathcal{A} \otimes \mathcal{B} = (|\mathcal{A}| \otimes |\mathcal{B}|, (\mathcal{A} \otimes \mathcal{B})_p, (\mathcal{A} \otimes \mathcal{B})_{cp}) \quad \text{where}$$
$$\begin{aligned} (\mathcal{A} \otimes \mathcal{B})_p &= \{\alpha \otimes \beta | \alpha \in \mathcal{A}_p, \beta \in \mathcal{B}_p\} \\ (\mathcal{A} \otimes \mathcal{B})_{cp} &= \mathbf{G}\mathcal{C}(\mathcal{A}, \mathcal{B}^\perp) \end{aligned}$$

Note that this last equality makes sense, because:

$$\mathbf{G}\mathcal{C}(\mathcal{A}, \mathcal{B}^\perp) \subseteq \mathcal{C}(|\mathcal{A}|, |\mathcal{B}|^\perp) \cong \mathcal{C}(|\mathcal{A}| \otimes |\mathcal{B}|, \mathbf{1}^\perp)$$

We also define the unit for the tensor product by $\mathbf{1_G} = (\mathbf{1}, \{id_\mathbf{1}\}, \mathcal{C}(\mathbf{1}, \mathbf{1}))$

We define linear negation by the formula:

$$\mathcal{A}^\perp = (|\mathcal{A}|^\perp, \mathcal{A}_{cp}, \mathcal{A}_p)$$

It is straightforward to verify that these definitions give a symmetric monoidal category and $(\ )^\perp$ defines a contravariant, involutive functor with the appropriate properties. Thus $\mathbf{G}\mathcal{C}$ is $*$-autonomous. □

We remark that in a logical setting one can think of an object $\mathcal{A} \in \boldsymbol{G}\mathcal{C}$ as a formula A in $\mathcal{C}$ together with a collection of proofs of A (the set $\mathcal{A}_p$) and a collection of refutations of A (the set $\mathcal{A}_{cp}$).

Proposition 3.11 (Tan) $\mathbf{G}\mathcal{C}$ *validates the* Mix *rule if and only if* $\mathcal{C}(\mathbf{1}, \mathbf{1}) = \{id_\mathbf{1}\}$. *We also note that* $\mathbf{G}\mathcal{C}(\mathbf{1_G}, \mathcal{A}) \cong \mathcal{A}_p$ *and* $\mathbf{G}\mathcal{C}(\mathcal{A}, \perp) \cong \mathcal{A}_{cp}$, *where* $\perp = (\mathbf{1_G})^\perp$ *is the dualizing object. Finally, the evident forgetful functor* $|\ \ |: \mathbf{G}\mathcal{C} \to \mathcal{C}$ *is $*$-autonomous, and has left and right adjoints.*

Observe from this that $\mathbf{GCoh_n}$ does not satisfy Mix, for $2 \leq n \leq \omega$; in particular this includes $\mathbf{Rel}, \mathbf{Coh},$ and $\mathbf{HCoh}$ (using $n = 2, 3, \omega$, respectively).

Definition 3.12 (Products in $\mathbf{GCoh_n}$ with $2 \leq n \leq \omega$) $\mathbf{GCoh_n}$ becomes a $*$-autonomous category with products and coproducts, given by the following:

(product:)

$$\mathcal{A}\&\mathcal{B} = (|\mathcal{A}|\&|\mathcal{B}|, (\mathcal{A}\&\mathcal{B})_p, (\mathcal{A}\&\mathcal{B})_{cp})$$

where

$$\begin{array}{rcl} |\mathcal{A}|\&|\mathcal{B}| & & \textit{is the product in } \mathbf{Coh}_n \\ (\mathcal{A}\&\mathcal{B})_p & = & \{\alpha + \beta \mid \alpha \in \mathcal{A}_p \text{ and } \beta \in \mathcal{B}_p\} := \mathcal{A}_p \frown \mathcal{B}_p \\ (\mathcal{A}\&\mathcal{B})_{cp} & = & \mathcal{A}_{cp} + \mathcal{B}_{cp} \end{array}$$

(coproduct:)

$$\mathcal{A} \oplus \mathcal{B} = (|\mathcal{A}| \oplus |\mathcal{B}|, (\mathcal{A} \oplus \mathcal{B})_p, (\mathcal{A} \oplus \mathcal{B})_{cp})$$

where

$$\begin{array}{rcl} |\mathcal{A}| \oplus |\mathcal{B}| & & \textit{is the coproduct in } \mathbf{Coh}_n \\ (\mathcal{A} \oplus \mathcal{B})_p & = & \mathcal{A}_p + \mathcal{B}_p \\ (\mathcal{A} \oplus \mathcal{B})_{cp} & = & \{\alpha + \beta \mid \alpha \in \mathcal{A}_{cp} \text{ and } \beta \in \mathcal{B}_{cp}\} := \mathcal{A}_{cp} \frown \mathcal{B}_{cp} \end{array}$$

Note that when $n = 2$ we have the products and coproducts of **GRel** which is $\mathbf{Coh}_2$ (cf. Definition 2.13).

3.3 MLL full completeness of $\mathbf{GCoh_n}$ with $2 < n \leq \omega$

We apply Hyland-Tan's double gluing construction to $\mathbf{Coh_n}$ to obtain $\mathbf{GCoh_n}$ with $2 < n \leq \omega$. In this section we shall observe that the category $\mathbf{GCoh_n}$ is fully complete for MLL (*without* Mix).

Lemma 3.13 *For an arbitrary $*$-autonomous category $\mathcal{C}$, the forgetful functor $|\ \ |: \mathbf{G}\mathcal{C} \to \mathcal{C}$ induces a canonical faithful functor*

$$\mathcal{I}\colon \textit{Dinat-}\mathbf{G}\mathcal{C} \longrightarrow \textit{Dinat-}\mathcal{C}$$

This functor preserves the $*$-autonomous structure with (co)products.

Proof. Given a dinatural transformation $\rho := \{\rho_{\mathcal{E}} : A(\mathcal{E};\mathcal{E}) \to B(\mathcal{E};\mathcal{E})\}$ in $\mathbf{G}\mathcal{C}$, let us apply the functor $|\ \ |$. Then we have a family $|\rho| := \{|\rho_{\mathcal{E}}| : A(|\mathcal{E}|;|\mathcal{E}|) \to B(|\mathcal{E}|;|\mathcal{E}|)\}$ of $\mathcal{C}$-morphisms. Recall that $|\mathcal{E}| = |\mathcal{F}|$ implies $\rho_{\mathcal{E}} = \rho_{\mathcal{F}}$ in $\mathbf{G}\mathcal{C}$ (cf. Theorem 1.3.2 [34]), and thus the family determines a family $|\rho| := \{|\rho|_E := \rho_{\mathcal{E}} : A(|\mathcal{E}|;|\mathcal{E}|) \to B(|\mathcal{E}|;|\mathcal{E}|)$ where $E = |\mathcal{E}|\}$ of morphisms indexed by the $\mathcal{C}$-objects. The dinaturality of the family is checked by using the fullness of $|\ \ |$. Hence we have a mapping from dinats of $\mathbf{G}\mathcal{C}$ to those of $\mathcal{C}$. Faithfulness of the functor is automatic, as is the fact that all structure is preserved. □

Lemma 3.14 *There is a canonical faithful functor*

$$\textit{Dinat-}\mathbf{GCoh_n} \longrightarrow \textit{Dinat-}\mathbf{GCoh}$$

Proof. This mapping is determined as the unique mapping making the following diagram commute. The vertical arrows are the faithful mappings of Lemma 3.13 and the lower horizontal arrow is the faithful mapping of Proposition 3.6:

$$\begin{array}{ccc} Dinat\text{-}\mathbf{GCoh_n} & \longrightarrow & Dinat\text{-}\mathbf{GCoh} \\ \downarrow & & \downarrow \\ Dinat\text{-}\mathbf{Coh_n} & \longrightarrow & Dinat\text{-}\mathbf{Coh} \end{array}$$

□

The following is the main lemma necessary for this subsection.

Lemma 3.15 *The forgetful functor* $|\quad|: \mathbf{Coh} \to \mathbf{Rel}$ *induces a canonical faithful functor*

$$Dinat\text{-}\mathbf{GCoh} \longrightarrow Dinat\text{-}\mathbf{GRel}$$

Proof. First the functor $|\quad|$ induces a functor $\restriction: \mathbf{GCoh} \to \mathbf{GRel}$ as follows: For each **GCoh**-object $\mathcal{E} = (E, \mathcal{E}_p, \mathcal{E}_{cp})$ with $E \in \mathbf{Coh}$, $\mathcal{E}_p \subseteq \mathbf{Coh}(\mathbf{1}, E)$ and $\mathcal{E}_{cp} \subseteq \mathbf{Coh}(E, \mathbf{1})$, we define the **GRel**-object $\mathcal{E}\restriction := (|E|, \mathcal{E}_p, \mathcal{E}_{cp})$. This is well defined since $\mathbf{Coh}(E, F) \subseteq \mathbf{Rel}(|E|, |F|)$. Second, given a dinatural transformation $\rho := \{\rho_{\mathcal{E}} : A(\mathcal{E}; \mathcal{E}) \to B(\mathcal{E}; \mathcal{E})\}$ in **GCoh**, let us apply the functor $\restriction$. Then we have the family $\rho\restriction := \{\rho_{\mathcal{E}} : A(\mathcal{E}\restriction; \mathcal{E}\restriction) \to B(\mathcal{E}\restriction; \mathcal{E}\restriction)\}$ of **GCoh**-morphisms. It can be checked that $\mathcal{E}\restriction = \mathcal{F}\restriction$ implies $\rho_{\mathcal{E}} = \rho_{\mathcal{F}}$ by using Lemma 3.5 and Lemma 3.13. Hence $\rho\restriction$ determines a family of morphisms indexed by **GRel**-objects. Dinaturality of the family is a consequence of the fullness of the functor $\restriction$. Hence we have the mapping in the assertion. Faithfulness is automatic. □

Tan [34] proved the following full completeness result which indeed preceded the full completeness for **Coh** we have referred to in Fact 3.2:

Fact 3.16 (Tan [34]) *Dinat*-**GRel** *is fully complete for* MLL.

As a direct consequence of Fact 3.16 together with Lemma 3.14 and Lemma 3.15, we have

Proposition 3.17 (MLL full completeness)
For $2 < n \leq \omega$, *Dinat*-$\mathbf{GCoh_n}$ *is fully complete for* MLL.

3.4 Lifting Softness from HCoh to *Dinat*-HCoh

In this final subsection, we shall observe that the property of softness is preserved in the construction of *Dinat*-**HCoh** from **HCoh**.

Note first that softness of $\mathcal{C}$ does not necessarily imply softness of *Dinat*-$\mathcal{C}$. Given a dinat $\rho_{\underline{X}} : \mathbf{1} \longrightarrow (E_{1,1}(\underline{X}; \underline{X}) \oplus E_{1,2}(\underline{X}; \underline{X})) \parr \cdots \parr (E_{m,1}(\underline{X}; \underline{X}) \oplus E_{m,2}(\underline{X}; \underline{X}))$, softness of $\mathcal{C}$ implies that for each vector of objects $\underline{A} \in \mathcal{C}^n$, an instantiation $\rho_{\underline{A}}$ factors through some coproduct injection, the particular component however may depend on $\underline{A}$.

The categories $\mathbf{Coh_n}$ are $*$-autonomous categories with products such that the $\otimes$ unit $\mathbf{1}$ coincides with $\parr$ unit $\bot$, hence in particular $\mathbf{Coh_n}$ satisfies Mix . In this case, *Dinat*-$\mathbf{Coh_n}$ satisfies a slightly stronger property than m-ary softness: every dinat ρ of the following form factors through one of the $\oplus$;

$$\rho : \mathbf{1} \longrightarrow X^{\#}_{i_1} \parr \cdots \parr X^{\#}_{i_k} \parr (E_{1,1}(\underline{X}; \underline{X}) \oplus E_{1,2}(\underline{X}; \underline{X})) \parr \cdots \parr (E_{m,1}(\underline{X}; \underline{X}) \oplus E_{m,2}(\underline{X}; \underline{X}))$$

where $X_{i_j} (1 \leq j \leq k$ with $0 \leq k)$ is a variable from the list $\underline{X}$ and $X^{\#}_{i_j}$ is X_{i_j} or $X^{\bot}_{i_j}$, hence is a literal.

Let us call this notion *m-ary extended softness*. Extended softness is necessary for the proof of Full Completeness for ⅋ALL +Mix in Section 4.1.

Proposition 3.18 *Dinat-*$\mathbf{Coh_n}$ *is m-ary extended soft for all* $m < n$*, including* $n = \omega$*.*

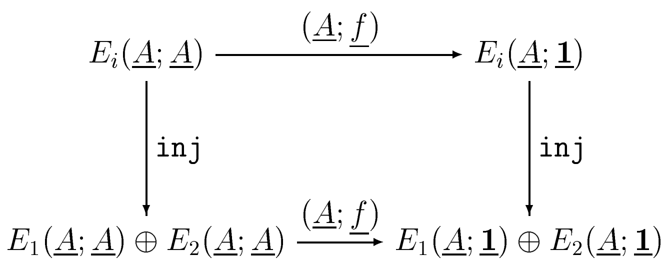
In particular (for the $n = \omega$ case above) we have:

Corollary 3.19 *Dinat-***HCoh** *is m-ary extended soft for all natural numbers m.*

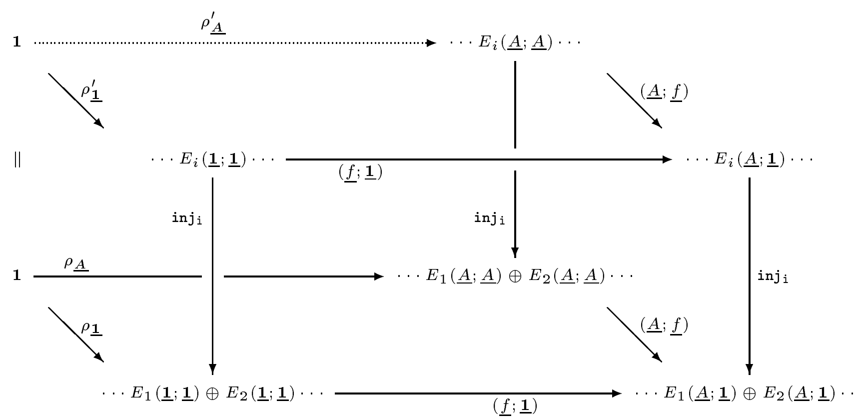
Proof (Proposition 3.18) . Given a dinat of the form

$$\rho : \mathbf{1} \longrightarrow X_{i_1}^{\#} ⅋ \cdots ⅋ X_{i_k}^{\#} ⅋ (E_{1,1}(\underline{X};\underline{X}) \oplus E_{1,2}(\underline{X};\underline{X})) ⅋ \cdots ⅋ (E_{m,1}(\underline{X};\underline{X}) \oplus E_{m,2}(\underline{X};\underline{X}))$$

and objects $\underline{A}$, consider an instantiation $\rho_{\underline{A}}$ as well as the instantiation $\rho_{\underline{\mathbf{1}}}$. Consider the morphism $\underline{f} : \underline{A} \to \underline{\mathbf{1}}$ induced from the morphisms $f_i : A_i \to \mathbf{1}$ given by $f_i = \{(a, \star) \mid a \in |A_i|\}$. We observe that in $\mathbf{Coh_n}$, the following diagram is a weak pullback, for all multivariate functors E_i, and for all $\underline{A}$:



Moreover, this is still a weak pullback if any MALL-definable functor is applied to this diagram. Softness, together with this weak pullback property, guarantees that $\rho_{\underline{\mathbf{1}}}$ factors through some coproduct injection; we shall show that this determines a coproduct injection for the entire dinatural ρ. Observe that, up to isomorphism, $\rho_{\underline{\mathbf{1}}} : \mathbf{1} \longrightarrow (E_{1,1}(\underline{\mathbf{1}};\underline{\mathbf{1}}) \oplus E_{1,2}(\underline{\mathbf{1}};\underline{\mathbf{1}})) ⅋ \cdots ⅋ (E_{m,1}(\underline{\mathbf{1}};\underline{\mathbf{1}}) \oplus E_{m,2}(\underline{\mathbf{1}};\underline{\mathbf{1}}))$, since $1^{\#}$ is either $\mathbf{1}$ or $\bot$, and in this model $\mathbf{1} = \bot$, which is the unit for ⅋ .



First by dinaturality of ρ with respect to $\underline{f}$, the bottom square of the diagram above commutes (we only indicate the specified components on objects; the remaining functorial type of ρ is denoted by $\cdots$.) Second softness of $\mathbf{Coh_n}$ implies that the instantiation $\rho_{\underline{\mathbf{1}}}$ factors through some coproduct injection, hence we have $\rho_{\underline{\mathbf{1}}} = \mathtt{inj}_i \circ \rho'_{\underline{\mathbf{1}}}$. By the previous remark, the right vertical square is a weak pullback. Moreover, the front square and the left vertical square commute. Hence, by the weak pullback property, $\rho_{\underline{A}}$ factors through some arrow $\rho'_{\underline{A}}$ as shown in the diagram. Thus $\rho_{\underline{A}}$ factors through the same coproduct component as $\rho_{\underline{\mathbf{1}}}$ does. Hence we have derived that the dinat ρ factors through a certain $\oplus$.

□

From now on, *softness* will always mean *extended softness* , since that is what is required in full completeness proofs.

4 Softness implies that dinats yield MALL-proof-structures

4.1 Full Completeness for ⅋ALL +Mix

Our purpose in this section is to prove that every dinatural transformation in **HCoh** (hence in particular **GHCoh**) corresponds to a Girard MALL proof-structure. For this we shall first prove that *Dinat*-**HCoh** is fully complete for the subsystem ⅋ALL +Mix. The subsystem ⅋ALL is obtained from MALL by restricting formulas and inference rules to the fragment not using the multiplicative connective $\otimes$ (in this formulation, we take ⅋ as primitive). Although the subsystem ⅋ALL +Mix is very elementary (in that only the one multiplicative connective ⅋ exists) full completeness for this subsystem is crucial to obtaining the main result in this subsection (Proposition 4.16).

Theorem 4.1 (softness implies ⅋ALL +Mix full completeness) *Suppose Dinat-$\mathcal{C}$ is soft and is fully complete for* MLL + Mix . *Then Dinat-$\mathcal{C}$ is fully complete for* ⅋ALL +Mix*; i.e., if Δ is a* ⅋ALL *sequent then every dinat $\rho : \mathbf{1} \to \Delta$ in $\mathcal{C}$ is a denotation of a* ⅋ALL +Mix *proof.*

In particular, by softness and multiplicative full completeness of *Dinat*-**HCoh** (see Corollary 3.19 and Proposition 3.7) we obtain:

Corollary 4.2 *Dinat*-**HCoh** *is fully complete for* ⅋ALL +Mix.

Proof (Theorem 4.1) . By induction on the number of additive connectives in Δ. Since every outermost occurrence of ⅋ in a formula occurring in Δ is replaced by a comma, we may assume by convention that every ⅋ALL sequent $\vdash \Delta$ is of the form $\vdash A_1, \ldots, A_n$, where for each i the outermost logical connective of A_i (if it exists) is additive or A_i is a literal.

(Base Case–no additive connectives)
Δ is of the form $\ell_1, \ldots, \ell_n$, where each ℓ_i is a literal. Note that this is an MLL sequent. Now the MLL+Mix full completeness in *Dinat*-$\mathcal{C}$ implies that Δ must be $p_1, p_1^\perp, \ldots, p_m, p_m^\perp$ and the ρ is the interpretation of a proof consisting of successively applying the Mix rule $(m-1)$-times to m axiom instances $\vdash p_1, p_1^\perp, \ldots, \vdash p_m, p_m^\perp$:

$$\dfrac{\vdash p_1, p_1^\perp \quad \vdash p_2, p_2^\perp}{\vdash p_1, p_1^\perp, p_2, p_2^\perp}\ \mathsf{Mix}$$

$$\ddots \qquad \vdots$$

$$\dfrac{\vdash p_1, p_1^\perp, \ldots, p_{m-1}, p_{m-1}^\perp \quad \vdash p_m, p_m^\perp}{\vdash p_1, p_1^\perp, \ldots, p_{m-1}, p_{m-1}^\perp, p_m, p_m^\perp}\ \mathsf{Mix}$$

(The case where Δ contains at least one additive connective)

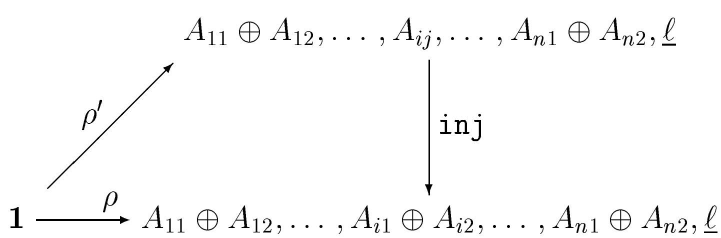
- (Case 1): If there exists a formula in Δ whose outer-most connective is &: Namely Δ is $\Delta_1, A_1\&A_2, \Delta_2$: Then by composing the projections with respect to this &, two dinats ρ_i are obtained with $i = 1, 2$ (note: projections are natural, so they compose with dinaturals):

$$\rho_i : \mathbf{1} \to \Delta_1, A_i, \Delta_2$$

  By the induction hypothesis, ρ_i is a denotation of a proof for $i = 1, 2$. Hence so is ρ because to obtain ρ from ρ_1 and ρ_2 corresponds to the following MALL inference

$$\frac{\vdash \Delta_1, A_1, \Delta_2 \quad \vdash \Delta_1, A_2, \Delta_2}{\vdash \Delta_1, A_1\&A_2, \Delta_2}\,\&.$$

- (Case 2): Negation of Case 1: all the outer-most connectives of the formulas (except literals) in Δ are $\oplus$. Then Δ is of the form $A_{11} \oplus A_{12}, \ldots, A_{n1} \oplus A_{n2}, \underline{\ell}$, where $\underline{\ell}$ denotes a sequence $\ell_1, \ldots, \ell_k$ of literal-types. Softness means that ρ factors through one of the $\oplus$'s; hence we obtain a factorization ρ' as follows:



  By the induction hypothesis, ρ' is a denotation of a proof, hence so is ρ because to obtain ρ from ρ' corresponds to the following MALL inference

$$\frac{\vdash A_{11} \oplus A_{12}, \ldots, A_{ij}, \ldots, A_{n1} \oplus A_{n2}, \underline{\ell}}{\vdash A_{11} \oplus A_{12}, \ldots, A_{i1} \oplus A_{i2}, \ldots, A_{n1} \oplus A_{n2}, \underline{\ell}}\,\oplus.$$

  In other words, the above $\oplus$-rule induces a natural transformation `inj` which composes with the dinatural ρ' to give the dinatural ρ.

□

4.2 Girard's MALL proof-structures

Now we recall the definition of multiplicative-additive proof-structure invented by Girard [18]:

Definition 4.3 (MALL proof-structure (cf. [18, 2, 32])) A *proof-structure* Θ consists of the following:

- Occurrences of formulas and links. Each occurrence of a link takes its premise(s) and conclusion(s) from among the formula occurrences and satisfies column (i) in the table below.

- A set of *eigenweights* $\{p_{L_1}, \ldots, p_{L_n}\}$ where $L_1, \ldots, L_n$ is the list of all &-links occurring in Θ and each p_{L_i} is a boolean variable associated with &-link L_i.
- For each occurrence A of a formula and occurrence L of a link, a *weight* $w(A)$ and a *weight* $w(L)$, each of which is a non-zero element in the boolean algebra generated by the eigenweights and satisfies column (ii) in the table below, as well as (iii) and (iv):

link L	**(i)** $L\frac{\text{premise(s)}}{\text{conclusion(s)}}$:	**(ii)** weights of L and its premise(s):
axiom-link	$\overline{A \quad A^{\perp}}$	
$\otimes$-link	$\frac{A \quad B}{A \otimes B}$	$w(L) = w(A) = w(B)$
$⅋$-link	$\frac{A \quad B}{A ⅋ B}$	$w(L) = w(A) = w(B)$
&-link	$\frac{A \quad B}{A \& B}$	$w(A) = p_L.w(L)$ and $w(B) = \neg p_L.w(L)$
$\oplus_1$-link	$\frac{A}{A \oplus B}$	$w(L) = w(A)$
$\oplus_2$-link	$\frac{B}{A \oplus B}$	$w(L) = w(B)$

(iii) $w(A) = \Sigma w(L)$ with L ranging over the links whose conclusion is A. Moreover the sum satisfies the *disjointness property*; i.e., if L_1 and L_2 are distinct links sharing the same conclusion A then $w(L_1).w(L_2) = 0$.
(iv) $w(A) = 1$ for a formula A which is not a premise of any link, i.e. which is a *conclusion* of Θ.

Moreover a proof-structure Θ satisfies the following two conditions:

dependency condition: Every weight of a formula and a link in Θ is a product of eigenweights and negations of eigenweights (up to boolean equivalence), i.e. is a *monomial.*

technical condition: For every weight v occurring in Θ and a &-link L, $v.\neg w(L)$ belongs to the boolean algebra generated by the eigenweights distinct from p_L.

Throughout the paper we take as convention that all monomial weights considered are *reduced*, i.e. that occurrences of $\epsilon p.\epsilon p$ (with $\epsilon \in \{1, \neg\}$) are replaced by ϵp and occurrences of $p.\neg p$ are replaced by 0. Under this convention we define

Definition 4.4 (Dependency) A (reduced) monomial weight w *depends on an eigenweight* p when ϵp appears in w with $\epsilon \in \{1, \neg\}$.

The following is a basic property of non-zero monomial weights:

Lemma 4.5 *For non-zero monomial weights v and w such that $0 \neq v \subseteq w$, if w depends on an eigenweight p then v also depends on p.*

Note that Lemma 4.5 cannot be extended to polynomial weights.

Girard's technical condition has also been examined by other authors. Let us summarize the known facts:

Remark 4.6 (Girard's technical condition) The following are equivalent to the technical condition:

(i.) Abramsky-Melliès [2]: For every weight v occurring in Θ, if v depends on p_L then $v \subset w(L)$.

(ii.) O. Laurent [32]: $w(L)$ does not depend on p_L and for every weight v occurring in Θ, if v depends on p_L then $v \subseteq w(L)$.

Remark 4.7 (Replacing $\otimes$ by $\parr$ in structures) If in a proof-structure, we choose a particular $\otimes$-link and we replace it by a $\parr$-link, and we replace all occurrences of $\otimes$ appearing hereditarily *below it* by $\parr$, then the resulting structure is still a proof-structure.

Finally, we would like to make an important remark on weight assignments for cut-free MALL structures.

Remark 4.8 (weights and additive links: softness of MALL p-s's) Each link in a cut-free MALL proof structure Θ corresponds to a unique connective occuring among the conclusions of Θ. However there may exist several links corresponding to any given connective in the conclusion, because of additive contractions. If a connective in a conclusion of Θ has several corresponding links hereditarily above it, their weights must all be strictly less than 1, since moving upwards in the structure, weights strictly decrease in additive contractions. Hence, if the weight of a link in Θ is 1, it is the *only* link corresponding to its namesake in the conclusion.

In fact, in Hamano [25] (cf. Proposition 1 of [25]), the following proposition is proved, as a consequence of Girard's technical condition: *An arbitrary cut-free proof structure has a link whose weight is 1.* The proposition is called *softness of* MALL *proof-structures* since it is shown to be a proof-structure counterpart of Joyal's categorical softness (see also Remark 4.15 below).

In Hamano [25] softness of MALL proof-structures is shown to imply the following sequentialization without $\otimes$.

Proposition 4.9 ($\parr$ALL +Mix sequentialization (cf. Hamano [25])) *Every* MALL *proof-structure without $\otimes$-links is* $\parr$ALL +Mix *sequentializable.*

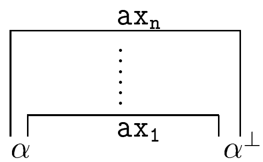
This proposition is later used in proving Lemma 4.14.

The proof structures arising from dinaturals will be shown to enjoy two distinguished properties introduced below (the unique link property and the no duplicate axiom-link property). These will be proved later in another subsection (cf. Corollary 4.44 and Corollary 4.55). These properties will be crucial to our full completeness theorem (in Section 6) which is based on **Rel**-like models.

Definition 4.10 (unique link and no duplicate axiom-link properties) A MALL proof-structure Θ is said to satisfy the *unique link property* and the *no duplicate axiom-link property* if the following hold respectively:

- **unique link property (UL):** If L in Θ is either a $\otimes$-link, $\parr$-link or &-link with conclusion D then it is the only link whose conclusion is that occurrence of D: i.e., there exist in Θ no distinct binary links whose conclusions are the same occurrence.

- **no duplicate axiom-link property (NDAL):** There occur in Θ no distinct axiom links $\mathtt{ax_1}, \mathtt{ax_2}, \ldots, \mathtt{ax_n}$ (with $n \geq 2$) whose (two) conclusions coincide and the sum of whose weights is 1, i.e. Θ has no occurrences of axiom links of the following form



$$\text{with} \sum_{i=1}^{i=n} w(\mathtt{ax_i}) = 1$$

A *UL* (respectively *NDAL*) proof-structure is a proof-structure which satisfies the unique link (respectively no duplicate axiom-link property) property.

In [18], Girard defines *sequentializable* MALL proof structures. His adequacy theorem states that to every MALL proof, we may associate a sequentializable proof-structure (see Remark 3 after Definition 5 of [18]). A delicate point is that the proof structure associated to a MALL proof is not necessarily unique.

We refer to Hamano [25] for an explicit algorithm for the adequacy theorem (Lemma 4.11 below) which yields the unique link property. This lemma will be crucial when we later show that every dinat in **HCoh** is associated with a proof-structure (see Corollary 4.55):

Lemma 4.11 (Adequacy theorem and UL (cf. Hamano [25])) *Every* MALL+Mix *proof π is interpreted by a* MALL+Mix *sequentializable proof structure Θ_π which satisfies the unique link property.*

Proof. If we take the largest *boundary* as defined in the proof of [25] to interpret &-inferences, the interpretation satisfies the property. □

This property will be mentioned again later in Lemma 4.34.

Remark 4.12 Neither MALL nor MALL+Mix sequentializable proof structures necessarily satisfy the unique link property. We emphasize again that this arises because the assignment of MALL proofs to MALL proof structures is not necessarily unique. This is quite different from what happens in the purely multiplicative case.

4.3 From dinats to MALL proof-structures

In this subsection we shall show how to construct MALL proof-structures from dinatural transformations on a soft category $\mathcal{C}$ which is MLL+Mix fully complete and whose Mix is monic (Proposition 4.16). This guarantees that every dinat in **HCoh**, hence in particular **GHCoh** , is associated with a MALL proof-structure (Corollary 4.55).

First, we recall the following Soundness Theorem [5, 22]:

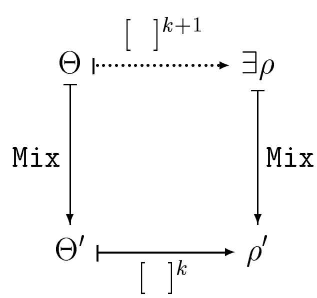
Lemma 4.13 (MALL+Mix soundness of the dinat interpretation) *Let $\mathcal{C}$ be an arbitrary $*$-autonomous category with products and coproducts, which satisfies* Mix. *Every* MALL+Mix *sequentializable proof structure Θ uniquely determines a dinatural transformation $[\Theta]$ of $\mathcal{C}$ such that $[\Theta]$ is a denotation of a MALL+Mix proof. This induces a mapping*

$$[-] : \text{MALL+Mix } \textit{Sequentializable Proof-Structures} \longrightarrow \textit{Dinat-}\mathcal{C}$$

Proof. We shall prove this by induction on the number of &-connectives in the conclusions of Θ.
(Base Case) This case is where the conclusions of Θ are an M⊕LL sequent. In this case Θ is identified with a unique cut-free MLL proof-structure, determined by the set of axiom-links, and these axiom-links uniquely determine a dinat of $\mathcal{C}$.
(Inductive Step) The case where some conclusions of Θ contain a &-connective. An important observation in this case is that, from *the softness of* MALL *proof-structures* (cf. Remark 4.8), Θ has a &-link whose weight is 1. Hence by Remark 4.8, this &-link must be the unique &-link corresponding to the & in the conclusion. Thus we shall denote by $\{\&_1, \ldots, \&_n\}$ the *non-empty* set of all &-links whose weights are 1: these each correspond to a unique and distinct namesake in the conclusion. If p_i denotes an eigenweight associated with the $\&_i$, the 2^n proof-structures $\Theta[p_1 = k_1, \ldots, p_n = k_n]$ with each $k_i \in \{0, 1\}$, are well-defined, indeed are MALL+Mix sequentializable. From the induction hypothesis, dinats $[\Theta[p_1 = k_1, \ldots, p_n = k_n]]$ are defined. We can uniquely define a dinat $[\Theta]$ from these dinats by the functoriality of the connectives binding the $\&_i$'s. The fact that $[\Theta]$ is actually a denotation of a MALL+Mix proof will be deferred to Example 4.29. □

The key point of this subsection is the following lifting lemma (Lemma 4.14) which follows from MALL+Mix Soundness for the dinatural interpretation for $\mathcal{C}$ where `Mix` is monic. We also require the observation that applications of `Mix` are commutative; i.e., the result of two applications of `Mix` to two distinct $\otimes$'s is unique and independent of the order of application. Categorically, this is a consequence of the naturality of the `Mix` morphism.

We first define a series of mappings $[\ \ ]^k$ by induction on natural numbers $k \geq 0$ so that each $[\ \ ]^{k+1}$ becomes an extension of $[\ \ ]^k$. For the base case, define $[\ \ ]^0$ to be $[\ \ ]$ from Lemma 4.13. Assume inductively that $[\ \ ]^k$ is well-defined, that Θ is a MALL proof structure and ρ is a dinat. Given $\Theta \notin Dom[\ \ ]^k$, we will say that $\Theta \in Dom[\ \ ]^{k+1}$ if (i) $\mathtt{Mix} \circ \Theta \in Dom[\ \ ]^k$ for some choice of a $\otimes$-link in Θ to which `Mix` is applied, and (ii) there exists a dinat ρ such that the type of ρ is that of Θ and $\mathtt{Mix} \circ \rho = [\mathtt{Mix} \circ \Theta]^k$. Since `Mix` is monic, ρ is unique if it exists. Hence for such a $\Theta \in Dom[\ \ ]^{k+1} \setminus Dom[\ \ ]^k$ satisfying (i) and (ii), we define $[\Theta]^{k+1} := \rho$. That is, the definition is described by the following commutative figure, where $\Theta' = \mathtt{Mix} \circ \Theta$ and $\rho' = \mathtt{Mix} \circ \rho$. Note that by construction, the types of Θ and ρ coincide:



Since applications of the monic `Mix` are commutative, $[\Theta]^{k+1}$ is well defined independently of the choice of $\otimes$-link to which `Mix` is applied.

Hence the above yields an extension $[\ \ ]^{k+1}$ of the mapping $[\ \ ]^k$ if we additionally demand that for $\Theta \in Dom[\ \ ]^k$, $[\Theta]^{k+1}$ is defined to be $[\Theta]^k$. In particular the domain of $[\ \ ]^{k+1}$ contains that of $[\ \ ]^k$ and is a certain subset of MALL proof-structures.

Second, we define the mapping $[\ \ ]^*$ as the union of the series $[\ \ ]^k$ of extensions: i.e., $[\Theta]^* := \rho$ whenever $[\Theta]^k = \rho$ for some $k \geq 0$. Thus we have defined the mapping $[\ \ ]^*$

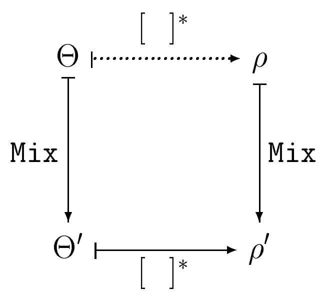
$$[\ \ ]^* : \text{A Certain Subset of } \mathsf{MALL} \text{ Proof-Structures} \longrightarrow \mathit{Dinat}\text{-}\mathcal{C}$$

Lemma 4.14 (Lifting of the dinat interpretation) *Let $\mathcal{C}$ be an arbitrary $*$-autonomous category with products and coproducts, which satisfies* `Mix`*, which we assume is monic. Then the mapping $[\ \ ]^*$ is a lifting (extension) of the interpretation $[-]$ of Lemma 4.13 such that the type of $[\Theta]^*$ is that of the p-s Θ and $[\ \ ]^*$ has the following property (†):*

(†) **lifting property of $[\ \ ]^*$ with respect to `Mix`:**

Let ρ and ρ' be a pair of dinats in $\mathcal{C}$ such that $\rho' = \mathtt{Mix} \circ \rho$ and let Θ and Θ' be a pair of proof-structures such that $\Theta' = \mathtt{Mix} \circ \Theta$ (this means that Θ' is obtained from Θ by a hereditary replacement of some $\otimes$-link (i.e. together with hereditary occurrences of the $\otimes$s) by $⅋$-links, in the sense of Remark 4.7. Then it follows that if $[\Theta']^* = \rho'$ and the type of Θ coincides with that of ρ, then $[\Theta]^* = \rho$.

We describe this property by the following commutative "figure":



where the right and left vertical arrows mean respectively $\rho' = \mathtt{Mix} \circ \rho$ and $\Theta' = \mathtt{Mix} \circ \Theta$.

In particular the property (†) implies the commutativity of $[\ \ ]^*$ and `Mix`*; i.e., it follows that $[\mathtt{Mix} \circ \Theta]^* = \mathtt{Mix} \circ [\Theta]^*$ for every Θ in the domain of $[\ \ ]^*$.*

Proof. This follows directly from the construction of $[\ \ ]^{k+1}$ from $[\ \ ]^k$ and the definition of $[\ \ ]^*$. □

Remark 4.15 ($[\ \ ]^*$ is not necessarily surjective) If *Dinat*-$\mathcal{C}$ is fully complete for MALL+Mix, then the lifting $[\ \ ]^*$ coincides with $[\ \ ]$ itself. But the converse is not true in general since the image of the mapping $[\ \ ]^*$ does not necessarily cover all the dinatural transformations of $\mathcal{C}$. For example, let $\mathcal{C} = \mathbf{Coh_n}$, for $n \neq \omega$. The n-ary Gustave dinaturals R mentioned in Proposition 3.8 show that $\mathbf{Coh_n}$ is not soft. On the other hand, Hamano [25] shows that *all* proof structures are soft, in the sense that a certain factorization/splitting property of MALL proof structures corresponds (under the mapping $[\ \ ]^*$) to softness of dinaturals. Hence, in general, the image of $[\ \ ]^*$ is soft, so the Gustave dinaturals cannot be in this image.

Continuing the above remark, if we impose additional conditions on $\mathcal{C}$, the interpretation $[\ \ ]^*$ above does indeed become surjective:

Proposition 4.16 (Every dinat has a weakly-associated proof-structure) *Let $\mathcal{C}$ be a $*$-autonomous category with products and coproducts, which satisfies* Mix *. Suppose $\mathcal{C}$ satisfies the following three conditions:*

(i) Dinat-$\mathcal{C}$ is soft.
(ii) Dinat-$\mathcal{C}$ is fully complete for MLL+Mix *.*
(iii) `Mix` *is monic in Dinat-$\mathcal{C}$.*

Then for every dinatural transformation ρ of $\mathcal{C}$, there exists a MALL *proof-structure Θ such that $\rho = [\Theta]^*$; that is, $[\ \ ]^*$ is surjective.*

In the above, Θ is referred to as a *weakly-associated proof-structure* to the dinat ρ.

Proof. By induction on the number of $\otimes$-connectives in the type of an arbitrarily given ρ.

(Base Case) The case where the type of ρ contains no $\otimes$: In this case the type of ρ is $\parr$ALL and the assertion follows from Theorem 4.1; that is, ρ is in the image of $[\ \ ]$.

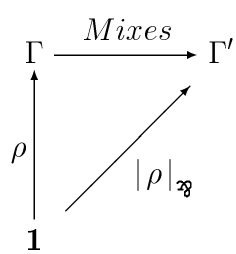
(Inductive Step) Choose one of the tensors in the type of ρ. Eliminate that tensor (replace it with a $\parr$ by composing with `Mix`) to obtain $\rho' := \mathtt{Mix} \circ \rho$. Then by the inductive hypothesis applied to ρ', there exists a proof-structure Θ' such that $\rho' = [\Theta']^*$. The proof-structure Θ, obtained by Remark 4.7, has type coinciding with that of ρ; moreover, it satisfies $\Theta' = \mathtt{Mix} \circ \Theta$. Then by property (†) of the map $[\ \ ]^*$, Θ is interpreted as the dinat $[\Theta]^*$ and we have

$$\begin{array}{rcll} \mathtt{Mix} \circ [\Theta]^* & = & [\mathtt{Mix} \circ \Theta]^* & \\ & = & [\Theta']^* & (\text{since } \Theta' = \mathtt{Mix} \circ \Theta) \\ & = & \rho' & (\text{since } \rho' = [\Theta']^*) \\ & = & \mathtt{Mix} \circ \rho & (\text{since } \rho' = \mathtt{Mix} \circ \rho). \end{array}$$

Thus $[\Theta]^* = \rho$, since `Mix` is monic in *Dinat-$\mathcal{C}$*. □

Let us examine the inductive step in Proposition 4.16 in more detail.

Remark 4.17 (Recovering Ξ_ρ from a sequentializable $\Xi_{|\rho|_{\parr}}$) Let Ξ_ρ denote a proof-structure as described in Proposition 4.16 such that $[\Xi_\rho]^* = \rho$. The following is an explicit algorithm for constructing such a Ξ_ρ. From the given dinat $\rho : \mathbf{1} \to \Gamma$, by composing with Mix maps, we obtain the dinat $|\rho|_{\parr}$ whose type is a $\parr$ALL sequent Γ' , where Γ' is obtained from Γ by replacing all of the occurrences of $\otimes$ by $\parr$. That is, if $\Gamma = \Gamma[A_{11} \otimes A_{12}, \ldots , A_{n1} \otimes A_{n2}]$ then $\Gamma' = \Gamma[A_{11} \parr A_{12}, \ldots , A_{n1} \parr A_{n2}]$. Define $|\rho|_{\parr}$, as the following $\parr$ALL dinat.



Thus by $\parr$ALL +Mix Full Completeness (Theorem 4.1), $|\rho|_{\parr}$ is a denotation of a proof. Thus by Lemma 4.13, a proof-structure $\Xi_{|\rho|_{\parr}}$ for $|\rho|_{\parr}$ is obtained. A proof-structure Ξ_ρ weakly-associated with ρ is obtained from $\Xi_{|\rho|_{\parr}}$ by replacing all occurrences of $\parr$-links and of the associated $⅋$'s, which are in the image of Mixes, by $\otimes$-links and $\otimes$, respectively. Note that Remark 4.7 ensures that the resulting structure Ξ_ρ is still a proof-structure. This proof-structure Ξ_ρ is often denoted by $\mathtt{Mix}^{-1} \circ \Xi_{|\rho|_{⅋}}$.

We define the set $WPS(\rho)$ *of proof structures weakly-associated with a dinat* ρ as follows:

$$WPS(\rho) := \{\Theta \mid \rho = [\Theta]^*\}$$

We shall later refine this to a non-empty subset $PS(\rho) \subseteq WPS(\rho)$ of *associated proof structures* (Definition 4.45). The latter will be shown to satisfy a fundamental property: a dinat ρ will denote a MALL proof iff all structures in $PS(\rho)$ are MALL proof nets (cf. the next subsection, and Corollary 4.53).

4.4 MALL Proof Nets

Next we recall Girard's sequentialization theorem [18] for proof-structures. A crucial step in the theorem was his introduction of the notion of *jumps* in a switching S, as defined below:

Definition 4.18 (Switching and Graphs of Additive Proof Structures (cf. [18, 2]))

- A *switching* S of a proof-structure Θ consists of the following three choices:
  (i) The choice of a *valuation* φ_S, which is a function from the set $\{p_{L_1}, \ldots, p_{L_n}\}$ of eigenweights to $\{0, 1\}$. φ_S induces a function from the weights of Θ to $\{0, 1\}$. The *slice* $sl(\varphi_S(\Theta))$ is obtained by restricting the proof structure Θ to the formula and link occurrences O such that $\varphi_S(w(O)) = 1$, i.e. we remove all formula and link occurrences in Θ whose weight under the valuation φ_S is 0.
  (ii) For each $⅋$-link L of $sl(\varphi_S(\Theta))$, a choice $S(L) \in \{l, r\}$.
  (iii) For each &-link L of $sl(\varphi_S(\Theta))$, a choice of a formula $S(L)$, called a *jump of* L, so that $S(L)$ is a conclusion of a link whose weight depends on p_L. A jump is *normal* if $S(L)$ is the premise A of L such that $\varphi_S(w(A)) = 1$. A *proper jump* is a jump which is not normal.
- A *normal switching* is a switching with no proper jump.
- For a switching S of a proof-structure Θ, the *graph* Θ_S is drawn as follows:
  - The vertices of Θ_S are the occurrences of the formulas of $sl(\varphi_S(\Theta))$.
  - For all axiom-links of $sl(\varphi_S(\Theta))$, we draw an edge between its conclusions.
  - For all $\oplus_i$-links of $sl(\varphi_S(\Theta))$, we drawn an edge between the conclusion and the premise.
  - For all $\otimes$-links of $sl(\varphi_S(\Theta))$, we drawn an edge between the conclusion and the left premise, and between the conclusion and the right premise.
  - For all $⅋$-links of $sl(\varphi_S(\Theta))$, we drawn an edge between the conclusion and the premise (left or right) selected by $S(L)$.
  - For all &-links of $sl(\varphi_S(\Theta))$, we drawn an edge between the conclusion and the jump $S(L)$ selected by S.

We will write $sl(\varphi(\Theta))$ for $sl(\varphi_S(\Theta))$ if S is clear from the context.

Remark 4.19 Let us make some remarks on slices.

1. A slice is a structure in which all additive links have now become unary. Thus, a slice can be identified with an MLL proof structure by erasing every (unary) additive link.

2. Following up on Remark 4.15, the interpretation $[\ \ ]^*$ inherits from $[\ \ ]$ the following property of commuting with valuations: for every valuation φ for Θ, $[sl(\varphi(\Theta))]^* = \varphi([\Theta]^*)$, where $\varphi([\Theta]^*)$ denotes the dinat resulting from $[\Theta]^*$ by composing with projections which are natural transformations (determined by φ).

3. If *Dinat-$\mathcal{C}$* is fully complete for MLL+Mix and a MALL proof-structure Θ is in the domain of the interpretation $[\ \ ]^*$, then every slice $sl(\varphi(\Theta))$ of Θ is a MLL+Mix proof-net by property 2 above.

Definition 4.20 (Proof nets) A *proof net* for MALL is a proof structure Θ such that Θ_S is acyclic and connected for every switching S. A *proof net* for MALL+Mix is a proof structure Θ such that Θ_S is acyclic for every switching S.

Proposition 4.21 (Sequentialization theorem for MALL (Girard [18])) *A* MALL *proof-structure is* MALL *sequentializable if and only if it is a* MALL *proof net.*

In [25] Hamano proved the following sequentialization theorem for MALL+Mix .

Proposition 4.22 (Sequentialization theorem for MALL + Mix ([25])) *A proof-structure is* MALL + Mix *sequentializable if and only if it is a* MALL+Mix *net.*

Indeed as a corollary of this MALL+Mix sequentialization theorem, a slightly stronger form of MALL sequentialization can be obtained:

Corollary 4.23 (cf. [25]) *A proof-structure Θ is a* MALL *proof-net if and only if (i) for every switching S the graph Θ_S is acyclic and (ii) for every normal switching S_0, the graph Θ_{S_0} is connected.*

Definition 4.24 (Associated normal switching) Let S be a switching for a proof structure. Associated to S there is a unique normal switching S_0 which agrees with S except all jumps in S_0 are normal (these are determined by φ_S). S_0 is called *the associated normal switching of S*.

From now on, S_0 will denote the associated normal switching of S.

Finally in this subsection, we have a lemma on weakly-associated proof-structures for a dinat. This lemma gives the fundamental connection between proof structures arising from dinats and proof nets.

Lemma 4.25 *A dinat ρ denotes a* MALL *proof iff the set of weakly associated proof structures $WPS(\rho)$ contains any proof net Θ.*

Proof. The only if part is direct: for a dinat ρ which is a denotation of a MALL proof, there exists a proof-net Θ such that $[\Theta] = \rho$.

As for the if part, suppose there exists $\Theta \in WPS(\rho)$ such that Θ is a proof-net, hence is sequentializable for MALL . Recall that $\rho = [\Theta]^*$ and $[\ \ ]^*$ is a lifting of $[\ \ ]$, as in Lemma 4.14. Note that in this case Θ is in the domain of $[\ \ ]$, thus we have $[\Theta]^* = [\Theta]$. This means that ρ is a denotation of a MALL proof by the soundness theorem, Lemma 4.13. □

4.5 Associated Proof-Structures

Let ρ be a dinat. The purpose of this subsection is to obtain a non-empty subset $PS(\rho) \subseteq WPS(\rho)$ of (strongly) associated proof-structures by adding a certain constraint on $WPS(\rho)$. The constrained class $PS(\rho)$ satisfies a strong soundness theorem: ρ denotes a MALL proof iff *all* elements of $PS(\rho)$ are proof-nets (Corollary 4.53). The class $PS(\rho)$ of associated proof structures will be important in the remainder of this paper.

The constraint we shall impose in forming $PS(\rho)$ from $WPS(\rho)$ is the notion of *legal total splittings* for a dinat $\mid \rho \mid_{\parr}$ [4] Total splittings are identified with a proof which the dinat denotes. There may be several syntactically different total splittings arising from one dinatural denotation; however legal total splittings yield our Fundamental Proposition and its Corollary 4.50, which states that our association of structures to dinats preserves cycles under semantical splittings. The Fundamental Proposition directly implies the soundness of the association (Corollary 4.53).

4.5.1 Semantical splittings of dinats

Definition 4.26 (semantical splittings of a dinat) For a dinat σ of MALL type, we define $\{\otimes, mix, \parr, \oplus, \&\}$*-splittings of* σ as follows:

- (Binary splittings): σ is split into two dinats σ_1 and σ_2 according to the following:
  - **$\otimes$-splitting:** If σ is written as $\sigma_1 \otimes \sigma_2$, then σ of type $\Delta_1, \Delta_2, A_1 \otimes A_2$ is split into dinats σ_i of type Δ_i, A_i with $i = 1, 2$.
  - **mix-splitting:** σ is written as σ_1 mix σ_2 (more simply as σ_1, σ_2), then σ of type Δ_1, Δ_2 is split into dinats σ_i of type Δ_i with $i = 1, 2$.
  - **&-splitting:** If σ is written as $\sigma_1 \& \sigma_2$, then σ of type $\Gamma, A_1 \& A_2$ is split into dinats σ_i of type Γ, A_i with $i = 1, 2$.
- (Unary splittings): σ is split into a single dinat σ_i for some $i \in \{1, 2\}$ according to the following:
  - **$\parr$-splitting:** If σ is written as $\parr(\sigma_1)$, then σ of type $\Gamma, A \parr B$ is split into a dinat σ_1 of type Γ, A, B.
  - **$\oplus_1$-splitting:** If σ is written as $\oplus(\sigma_1)$, then σ of type $\Gamma, A_1 \oplus A_2$ is split into a dinat σ_1 of type Γ, A_1.
  - **$\oplus_2$-splitting:** If σ is written as $\oplus(\sigma_2)$, then σ of type $\Gamma, A_1 \oplus A_2$ is split into a dinat σ_2 of type Γ, A_2.

That is to say, each splitting corresponds to the associated MALL + Mix rule.

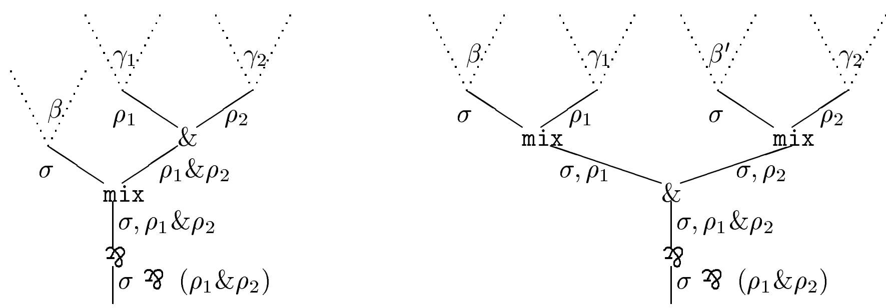
A *total* splitting of a dinat σ is a series of successive splittings so that no possible splitting is left to be done. A total splitting *terminates* if all the terminal dinats are identities on atoms.

Remark 4.27 (remarks on splittings) 1. Let $\mathcal{C}$ be an arbitrary $*$-autonomous category with products and coproducts, which satisfies `Mix`. For every $\mathcal{C}$-dinat ρ which denotes a MALL+Mix proof, all total splittings of ρ terminate; i.e. any successive iterations of $\{\otimes, mix, \parr, \oplus_1, \oplus_2, \&\}$-splittings of ρ yield a set of identity dinats.

[4] Recall that a proof-structure Ξ_ρ weakly associated with ρ is $\mathtt{Mix}^{-1} \circ \Xi_{|\rho|_{\parr}}$ (Remark 4.17) and that $|\rho|_{\parr}$ is a denotation of a proof, by $\parr$ALL +Mix full completeness (Theorem 4.1).

2. A total splitting α is represented as a tree, where each node corresponds to a splitting and where each edge attached to a node corresponds to the resulting dinat(s) after a splitting. The root of the tree represents the first splitting and the leaves of the tree represent the terminal dinats.

Example 4.28 (tree representation of splittings) The following are tree representations of two total splittings α and α' for a dinat $\sigma \parr (\rho_1 \& \rho_2)$:



Splitting α Splitting α'

In the above β, β', γ_1 and γ_2 are total splittings for σ, ρ_1 and ρ_2, respectively.

Example 4.29 ([Θ] is a proof) In Lemma 4.13 (the soundness of the dinat interpretation), our construction of [ ] ensures that [Θ] semantically splits, in a manner corresponding to a splitting of a terminal link of Θ. Moreover, the image of [ ] is closed under semantical splitting. Hence we have that the dinat [Θ] corresponds to a MALL+Mix proof.

As a special case of (1) of Remark 4.27, we have

Lemma 4.30 (Total splittings terminate for HCoh-dinats of ⅋ALL type) *For every* **HCoh**-*dinat σ of* ⅋ALL *type, total splittings of σ terminate; i.e. any successive iterations of $\{mix, \parr, \oplus_1, \oplus_2, \&\}$-splittings of σ yield a set of identity dinats.*

Proof. From the ⅋ALL +Mix full completeness of *Dinat*-**HCoh** of Corollary 4.2 □

We define a legal total splitting by imposing constraints on &-splittings as follows:

Definition 4.31 (legal total splitting) Let σ be a dinat of MALL type with a total splitting α. α is *legal* if the splittings in it satisfy the following constraints:

- Every &-splitting for a dinat occurring in α is executed under the proviso that it is impossible to subsequently execute any $\{\otimes, mix, \parr, \oplus_1, \oplus_2\}$-splittings to the dinat.

In terms of the tree representing α, the above constraints say that for every &-splitting node, the unique dinat attached to the node before the splitting cannot then be split by any further $\{\otimes, mix, \parr, \oplus_1, \oplus_2\}$-splittings.

Example 4.32 The total splitting α in Example 4.28 is legal (if β, γ_1 and γ_2 are). On the other hand, the total splitting α' is not legal: although the dinat $\sigma, \rho_1 \& \rho_2$ can be split via mix, instead a &-splitting of the dinat is executed first.

From Remark 4.27, we have the following:

Corollary 4.33 (Existence of legal total splittings) Let $\mathcal{C}$ be the same as in Remark 4.27 . For every $\mathcal{C}$-dinat ρ which denotes a MALL+Mix proof, there exists at least one legal total splitting.

4.5.2 Strongly Associating Proof Structures to Dinats

Our goal in this subsection is to improve Proposition 4.16 which says that under appropriate conditions on a category $\mathcal{C}$, a $\mathcal{C}$-dinat has a weakly associated proof structure. Indeed we completely characterize those $\mathcal{C}$-dinats that denote MALL proofs (Proposition 4.53). This involves, as we show in Corollary 4.55, that MALL proof structures associated with dinats on **HCoh** and **GHCoh** satisfy the UL and NDAL properties.

Let α denote a terminating total splitting for a dinat σ. Then every such α can be seen as a MALL+Mix proof which σ denotes. Of course, for a given dinat a total splitting α–even if one exists– is not uniquely determined. This corresponds to the fact that a dinat σ can denote several *syntactically different* proofs. We shall first show that for every such α, we can associate a canonical proof-structure $\Theta(\alpha)$ satisfying the unique link property and the no duplicate axiom link property. For this we begin with several lemmas and definitions.

First, we demonstrate the canonical interpretation of logical rules. This will ensure the unique link property (cf. Corollary 4.44).

Lemma 4.34 (Canonical proof structure interpretation of logical rules) *Suppose a* MALL+Mix *proof* π *is obtained from proof(s)* π_i *by means of a logical rule* $@ \in \{\otimes, mix, \parr, \oplus_1, \oplus_2, \&\}$*; i.e., the last inference of* π *is* @*. From any UL proof-structures* Θ_i *whose sequentializations are* π_i*, a canonical UL proof-structure is uniquely constructed such that its sequentialization is* π *and its splitting corresponding to* @ *yields the proof-structure(s)* Θ_i *(here, a splitting of such a proof structure is obtained by removing a terminal* @*-link).*

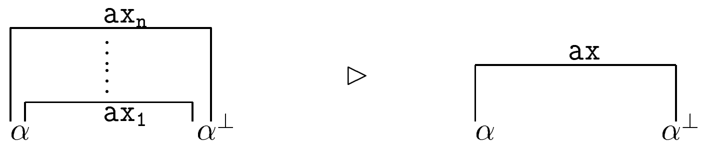
The proof-structure which we construct above is denoted by $\Theta_1 \otimes \Theta_2$, Θ_1 $\texttt{mix}$ Θ_2 (more simply Θ_1, Θ_2), $\parr(\Theta_1)$, $\oplus(\Theta_1)$, $\oplus(\Theta_2)$, or $\Theta_1 \& \Theta_2$, depending upon the choice of logical rule @.

Proof. We shall prove the case where @ is & (the other cases are trivial). The algorithm given in [25] to interpret &-inferences tells us how to merge two proof-structures Θ_1 and Θ_2 with the same context in the conclusions. Let us take the largest *boundary* among other boundaries, as defined in the proof. Note that the largest boundary is uniquely determined. Thus we canonically obtain a proof-structure $\Theta_1 \& \Theta_2$ for the assertion. □

Remark 4.35 The above lemma states that the canonical interpretation of logical rules preserves the unique link property. Note however that the canonical interpretation does not necessarily preserve the no duplicate axiom link property defined in Definition 4.10. This is why we introduce Definition 4.36 below.

Next we define a rewriting relation $\triangleright$ and demonstrate some of its properties; in particular, it will ensure the no duplicate axiom-link property in Corollary 4.44.

Definition 4.36 (rewriting to shrink duplicate axiom-links) Let us define a rewriting relation $\triangleright$ from duplicate axiom-links $\texttt{ax}_1, \texttt{ax}_2, \ldots, \texttt{ax}_n$ with $\sum_{i=1}^{i=n} w(\texttt{ax}_i) = 1$ into the single axiom-link $\texttt{ax}$ such that $w(\texttt{ax}) = 1$:



This is extended to a reduction relation $\rhd$ on all proof-structures.

Let us call a tuple $\mathtt{ax_1}, \mathtt{ax_2}, \ldots, \mathtt{ax_n}$ of axiom-links a *redex* for $\rhd$.

Lemma 4.37 (uniqueness of normal form wrt $\rhd$) *The normal form for a proof-structure under the reduction relation $\rhd$ is unique.*

Proof. First, observe that occurrences of redexes are uniquely determined in every proof-structure by virtue of the constraint that $\sum_{i=1}^{i=n} w(\mathtt{ax_i}) = 1$. Moreover, rewriting $\rhd$ does not give rise to any new redexes. $\square$

Lemma 4.38 (invariance of the interpretation $[\ \ ]^*$ under $\rhd$) *Suppose $[\Theta]^* = \rho$. If $\Theta \rhd \tilde{\Theta}$, then $[\tilde{\Theta}]^* = \rho$. That is to say the interpretation $[\ \ ]^*$ is invariant under reduction by $\rhd$.*

Remark 4.39 Lemma 4.38 ensures that one can apply $\rhd$ in a proof-structure associated with a dinat since an application preserves the interpretation $[\ \ ]$.

The previous lemmas allow us to obtain canonical UL and NDAL proof-structures corresponding to terminating total splittings:

Proposition 4.40 (canonical p-s for terminating splittings) *Let ρ be a dinat. Every terminating total splitting α for ρ is canonically interpreted by a unique* MALL+Mix *sequentializable UL and NDAL proof-structure $\Theta(\alpha)$ such that $[\,\Theta(\alpha)\,] = \rho$.*

The proof-structure $\Theta(\alpha)$ above, whose sequentialization is α, is called the *canonical proof-structure for terminating total splitting α.*

Proof. By induction on the size of α, for a dinat ρ. We shall prove the case where the first splitting of α is a &-splitting. This yields total splittings α_i for dinat ρ_i with $i \in \{1, 2\}$ (the other cases are trivial). By induction hypothesis α_i is interpreted by a structure $\Theta(\alpha_i)$ such that $[\Theta(\alpha_i)] = \rho_i$ with $i \in \{1, 2\}$ and $\Theta(\alpha_i)$ satisfies UL and NDAL.

First, from Lemma 4.34, we have a canonical UL proof-structure $\Theta(\alpha_1)\&\Theta(\alpha_2)$ such that $[\Theta(\alpha_1)\&\Theta(\alpha_2)] = \rho$. Note that $\Theta(\alpha_1)\&\Theta(\alpha_2)$ may have duplicate axiom-links even if the individual $\Theta(\alpha_i)$ are NDAL proof-structures (cf. Remark 4.35) .

Second, by Lemma 4.37, $\Theta(\alpha_1)\&\Theta(\alpha_2)$ is uniquely reducible to a proof-structure, say $\Theta(\alpha)$; i.e., $\Theta(\alpha_1)\&\Theta(\alpha_2) \rhd_* \Theta(\alpha)$, where $\rhd_*$ is the reflexive transitive closure of $\rhd$. By virtue of Lemma 4.38, we obtain that $[\Theta(\alpha)] = \rho$. $\square$

By using the notion of canonical proof-structures of Proposition 4.40, we are now ready to define the following:

Definition 4.41 (restricting $[\ \ ]$ to $[\ \ ]_-$)

We restrict the mapping $[\ \ ]$ of Lemma 4.13 to the the mapping $[\ \ ]_-$ by restricting Θ to only structures given by legal total splittings, i.e.

$$\Theta \in Dom[\ \ ]_- \quad \text{iff} \quad \Theta = \Theta(\alpha) \text{ for some legal total splitting } \alpha \text{ of the dinat } [\Theta].$$

Since for any Ξ in the domain of $[\ ]$, $[\Xi]$ denotes a MALL+Mix proof, there exists at least one legal total splitting α for the dinat $[\Xi]$ (cf. Remark 4.33), hence $[\Theta(\alpha)]_{-} = [\Xi]$. This implies that the image of $[\ ]_{-}$ coincides with that of the original $[\ ]$.

Lemma 4.42 (lifting $[\ ]^{*}_{-}$ of $[\ ]_{-}$) The interpretation $[\ ]_{-}$ has lifting $[\ ]^{*}_{-}$ as in Lemma 4.14. Then $[\ ]^{*}_{-}$ becomes a restriction of $[\ ]^{*}$.

Remark 4.43 If a proof-structure Θ is in the domain of $[\ ]^{*}_{-}$, then it satisfies the unique link property and the no duplicate axiom link property by Proposition 4.40.

With this remark, Proposition 4.16 of the previous subsection directly implies the following:

Corollary 4.44 (Every associated p-s for a dinat satisfies UL and NDAL) *Let $\mathcal{C}$ be a $*$-autonomous category with products and coproducts, which satisfies* Mix *. Suppose $\mathcal{C}$ satisfies the following three conditions:*

(i) Dinat-$\mathcal{C}$ is soft.
(ii) Dinat-$\mathcal{C}$ is fully complete for MLL+Mix *.*
(iii) `Mix` *is monic in Dinat-$\mathcal{C}$.*

Then for every dinatural transformation ρ of $\mathcal{C}$, there exists a MALL *proof-structure Θ such that $\rho = [\Theta]^{*}_{-}$. Every such Θ satisfies the unique link property and the no duplicate axiom-link property. Θ is said to be an* associated proof-structure *for a dinat ρ.*

Proof. The proof is similar to Proposition 4.16 by noting the following for each case: (Base Case) The image of $[\ ]_{-}$ coincides with that of $[\ ]$; then we apply Remark 4.43. (Inductive Step) The properties UL and NDAL are preserved under replacement of a $\parr$-link by a $\otimes$-link. □

Until the end of this subsection, let $\mathcal{C}$ denote any category satisfying (i), (ii) and (iii) of Corollary 4.44, hence in particular **HCoh**. Using Corollary 4.44, we can now define the non-empty set $PS(\rho)$ of proof-structures (strongly) associated to a dinat ρ.

Definition 4.45 (strongly associated proof-structures) Let ρ be a dinat of $\mathcal{C}$. We define

$$PS(\rho) := \{\Theta \mid \rho = [\Theta]^{*}_{-}\}$$

By Remark 4.17, which gave a direct algorithm to define $[\ ]^{*}_{-}$, it may be equivalently defined by

$$PS(\rho) = \mathtt{Mix}^{-1} \circ PS(|\rho|_{\parr})$$

Since $|\rho|_{\parr}$ is a denotation of a $\parr$ALL+Mix proof by Theorem 4.1, $PS(|\rho|_{\parr})$ in the above may be explicitly described by

$$PS(|\rho|_{\parr}) = \{\ \Theta(\alpha) \mid\ \alpha \text{ is a legal total splitting for } |\rho|_{\parr}\ \}$$

First we note that $PS(\rho)$ is a nonempty subset of $WPS(\rho)$ since $[\ ]^{*}_{-}$ is a restriction of $[\ ]^{*}$ and the images of $[\ ]^{*}_{-}$ and $[\ ]^{*}$ coincide. Second, note that all proof-structures $\Theta \in PS(\rho)$ satisfy the unique link property and the no duplicate axiom-link property by Remark 4.43.

We shall refer to elements of $PS(\rho)$ as *associated proof structures* when the meaning is clear. We automatically have the following lemma, corresponding to Lemma 4.25 of the previous subsection:

Lemma 4.46 *A dinat ρ denotes a* MALL *proof iff $\exists\Theta \in PS(\rho)$ Θ is a proof-net.*

In the next subsection, we shall considerably strengthen this Lemma.

4.5.3 Soundness of associated proof structures

Our motivation for imposing legality in defining $[\ \ ]_{-}$ (hence to its lifting $[\ \ ]^{*}_{-}$) is to obtain a much stronger proposition (Corollary 4.53 below) than Lemma 4.46 above: this will guarantee that ρ denotes a MALL proof iff *all* elements of $PS(\rho)$ are proof nets.

We begin by a more detailed analysis of splittings of dinats, which we call Fundamental Proposition.

Proposition 4.47 (Fundamental Proposition) *Suppose that a $\mathcal{C}$-dinat ρ can be split via a @-splitting with $@ \in \{\otimes, mix, ⅋, \oplus_1, \oplus_2, \&\}$. Then every $\Theta \in PS(\rho)$ has the corresponding @-splitting.*

Proof. We shall prove the assertion by induction on the number of &-connectives in the type of dinat ρ.
(Base Case–no &-connectives)
The assertion is obvious since in this case ρ is identified with a multiplicative dinat.

(Induction Case)
The assertion is obvious for a splitting $@ \in \{⅋, \oplus_1, \oplus_2, \&\}$ since by virtue of the unique link property of Θ, the corresponding @-link in Θ is terminal and every terminal $\{⅋, \oplus_1, \oplus_2, \&\}$-link can be split. Thus we shall prove the assertion for ρ that are split into two dinats ρ_1 and ρ_2 via $@ \in \{mix, \otimes\}$.

First we recall, from Definitions 4.45, that Θ is of the form $\mathtt{Mix}^{-1} \circ \Theta(\alpha)$ with $|\,\Theta\,|_{⅋} = \Theta(\alpha) \in PS(|\,\rho\,|_{⅋})$ for a certain legal total splitting α for $|\,\rho\,|_{⅋}$.

In the following Cases 1 and 2, legality of α plays a crucial role. For these cases we introduce some terminology as follows. Recall from Remark 4.27 (2), that we identify α with a tree. We say that a dinat *appears in* α if it appears in some edge of the tree α. We say that appearances of dinats in α are *independent* if the subtrees determined by the corresponding edges of α are disjoint.

(Case 1) The case where ρ splits via mix; in this case ρ can be written as ρ_1, ρ_2 by making the splitting explicit. Note first that the dinat $|\,\rho\,|_{⅋}$ is $|\,\rho_1\,|_{⅋}, |\,\rho_2\,|_{⅋}$, hence can also be split into $|\,\rho_1\,|_{⅋}$ and $|\,\rho_2\,|_{⅋}$ via mix. Since the total splitting α for $|\,\rho\,|_{⅋}$ is legal, we have the following:

Observation: There exist sets $\{\sigma_{1i}\}_{i\in I}$ and $\{\sigma_{2j}\}_{j\in J}$ of dinats satisfying (i) and (ii):

(i) Each of σ_{1i} and σ_{2j} appears in the total splitting α and all appearances $\{\sigma_{1i}, \sigma_{2j}\}_{i\in I, j\in J}$ are independent.

(ii) Each σ_{1i} (respectively σ_{2j}) is obtained from $|\,\rho_1\,|_{⅋}$ (respectively $|\,\rho_2\,|_{⅋}$) by a series of splittings without any use of &-splittings.

From the observation, it holds that the proof-structure $|\,\Theta\,|_{\wp}$ is a union of two proof-structures $\Xi_1 \in PS(|\,\rho_1\,|_{\wp})$ and $\Xi_2 \in PS(|\,\rho_2\,|_{\wp})$. Thus we conclude that the proof-structure Θ is a union of two proof-structures $\Theta_1 \in PS(\rho_1)$ and $\Theta_2 \in PS(\rho_2)$, where $\Theta_i := \mathtt{Mix}^{-1} \circ \Xi_i$ with $i \in \{1, 2\}$. Thus Θ has the corresponding mix splitting.

(Case 2) The case where ρ splits via $\otimes$; in this case ρ can be written as $\rho_1 \otimes \rho_2$ by making the splitting explicit: Note first that $|\,\rho\,|_{\wp}$ is $|\,\rho_1\,|_{\wp} \wp |\,\rho_2\,|_{\wp}$, hence can be split into $|\,\rho_1\,|_{\wp}$ and $|\,\rho_2\,|_{\wp}$ via mix (following a $\wp$-splitting). Since the total splitting α for $|\,\rho\,|_{\wp}$ is legal, we have the same observation as in the above Case 1. From the observation, it holds that the proof-structure $|\,\Theta\,|_{\wp}$ is a union of two proof-structures $\Xi_1 \in PS(|\,\rho_1\,|_{\wp})$ and $\Xi_2 \in PS(|\,\rho_2\,|_{\wp})$ by drawing the terminal $\wp$-link corresponding to the $\wp$-splitting. Thus we conclude that the proof-structure Θ is a union of two proof-structures $\Theta_1 \in PS(\rho_1)$ and $\Theta_2 \in PS(\rho_2)$ by drawing the terminal $\otimes$-link corresponding to the $\otimes$-splitting, where $\Theta_i := \mathtt{Mix}^{-1} \circ \Xi_i$ with $i \in \{1, 2\}$. Thus Θ has the corresponding $\otimes$-splitting. □

The reason why we have imposed the constraint of "legality" is to obtain this Proposition 4.47. The proposition need not be valid when $PS(\rho)$ is replaced by the bigger set of weakly associated proof structures $WPS(\rho)$, as follows:

Example 4.48 (Why legality is necessary) From the above Example 4.28, we define total splittings $\bar{\alpha}$ and $\bar{\alpha}'$ for a dinat $\sigma, \rho_1 \& \rho_2$ to be α and α' respectively without the first $\wp$-splitting. Then $\bar{\alpha}$ is legal but $\bar{\alpha}'$ is not legal, as explained in the example. Suppose that dinats σ and ρ_i ($i = 1, 2$) are denotations for a MALL+Mix proof. By observing that a dinat $\sigma, \rho_1 \& \rho_2$ can be split via mix into two dinats σ and $\rho_1 \& \rho_2$, we have the following:

(i) A p-s $\Theta(\bar{\alpha}) \in PS(\sigma, \rho_1 \& \rho_2)$ has the corresponding mix splitting.

(ii) On the contrary, a p-s $\Theta(\bar{\alpha}') \in WPS(\sigma, \rho_1 \& \rho_2)$ may not be correspondingly split via mix.

Since (i) is an example of Proposition 4.47, we shall explain (ii). First, a disjoint union $\Theta(\beta), \Theta(\gamma_1)$ (respectively $\Theta(\beta'), \Theta(\gamma_2)$) of $\Theta(\beta)$ (respectively $\Theta(\beta')$) and $\Theta(\gamma_1)$ (respectively $\Theta(\gamma_2)$) is an element of $WPS(\sigma, \rho_1)$ (respectively of $WPS(\sigma, \rho_2)$). Second, $\Theta(\bar{\alpha}')$ is obtained from these two unions via the canonical interpretation of the &-inference of Lemma 4.34; i.e., $\Theta(\bar{\alpha}')$ is $(\Theta(\beta), \Theta(\gamma_1)) \& (\Theta(\beta'), \Theta(\gamma_2))$. Then from the definition of &-interpretation, $\Theta(\bar{\alpha}')$ becomes a union of two (*not necessarily proof-*) structures: One is a proof-structure $\Theta(\gamma_1) \& \Theta(\gamma_2)$. The other is a superposition (arising in the &-interpretation) of two proof-structures $\Theta(\beta)$ and $\Theta(\beta')$ which share the same conclusions. It is important to observe that the latter structure is *not necessarily a proof-structure* without the guarantee of $\Theta(\beta) = \Theta(\beta')$, since there may occur, in superposing $\Theta(\beta)$ and $\Theta(\beta')$, a link whose weight depends on the eigenweight p associated with the &. Thus we conclude that $\Theta(\bar{\alpha}')$ need not have the corresponding mix splitting.

As a direct corollary of Proposition 4.47, we have the following Corollary on preservation of cycles:

Notation 4.49 We say that a proof-structure Θ *has a cycle* $\mathtt{C}$ if $\mathtt{C}$ appears in Θ_S under some switching S. We say that a dinat ρ *yields a cycle* $\mathtt{C}$ if there exists a proof-structure $\Theta \in PS(\rho)$ such that Θ has a cycle $\mathtt{C}$.

Corollary 4.50 (Preservation of Cycles) *Suppose that a $\mathcal{C}$-dinat ρ can be split into dinats ρ_i by means of a unary or binary rule. If ρ yields a cycle* $\mathtt{C}$*, then there exists $i \in \{1, 2\}$ such that ρ_i yields the cycle* $\mathtt{C}$.

Proof. Suppose that a dinat ρ can be split via a @-splitting. Suppose moreover, a cycle $\mathtt{C}$ appears in a proof-structure $\Theta \in PS(\rho)$. From Proposition 4.47, Θ can be correspondingly split via @ into Θ_i. Hence the cycle $\mathtt{C}$ is retained in some Θ_i with $i \in \{1, 2\}$. Since $\Theta_i \in PS(\rho_i)$, we have derived the assertion. □

Example 4.51 As an example of Corollary 4.50, let us consider the case where ρ can be split into ρ_1 and ρ_2 by means of a $\otimes$-splitting. In this case Proposition 4.47 (Fundamental Proposition) means that the proof-structure Θ is a union of two proof-structures $\Theta_1 \in PS(\rho_1)$ and $\Theta_2 \in PS(\rho_2)$ by drawing the terminal $\otimes$-link corresponding to the $\otimes$-splitting. This in particular means that for any &-link, say $\&_p$, occurring in Θ_1 (respectively, in Θ_2), no weight occurring in Θ_2 (respectively, Θ_1) depends on p. Hence no jump can be drawn between Θ_1 and Θ_2. Thus every path between a formula occurrence in Θ_1 and one in Θ_2 must go through the $\otimes$-link. Hence we conclude that if Θ has a cycle $\mathtt{C}$, then $\mathtt{C}$ must exist either in Θ_1 or Θ_2.

Remark 4.52 (structural preservation of cycles) Corollary 4.50 of the fundamental proposition states that our interpretation of dinat σ into the set $PS(\sigma)$ of proof-structures preserves cycles with respect to semantical splittings. Corollary 4.50 will be crucial later in obtaining Lemma 5.17, which will be used in the Main Theorem in Section 6.1.

From Corollary 4.50, we obtain the main result of this subsection:

Corollary 4.53 (Soundness of associated proof-structures) *A dinat ρ denotes a* MALL *proof iff $\forall\, \Theta \in PS(\rho)$, Θ is a proof-net.*

Proof. The "if" part is Lemma 4.46. Thus we shall prove the "only if" part. Note first that for a dinat ρ of MLL type, the assertion is obvious since $PS(\rho)$ is a singleton. Suppose we are given a dinat ρ denoting a MALL proof. From what we have just said, it holds that $\forall\, \Theta \in PS(\rho)\ \Theta_{S_0}$ is connected for all normal switchings S_0, since a normal switching yields a MLL dinat. Suppose for contradiction that $\exists\, \Theta \in PS(\rho)\ \Theta$ is not a proof-net. From Corollary 4.23 and the connectedness of a p-s under normal switchings, Θ must have a cycle. On the other hand, since ρ denotes a proof, there is a series of splittings for ρ which terminate. This implies from Corollary 4.50 that there arises an identity dinat which yields a cycle. This is a contradiction. □

Remark 4.54 Strictly speaking, Corollary 4.53 *together with* Lemma 4.46 is what is referred to as the soundness of associated proof-structures.

Now we arrive at an important consequence of this section:

Corollary 4.55 (MALL proof-structures associated with HCoh and GHCoh dinats) *Every dinatural transformation ρ of* **HCoh** *is associated with a set $PS(\rho)$ of UL and NDAL* MALL *proof-structures satisfying Lemma 4.46, Corollary 4.50 and Corollary 4.53. In particular, so is every dinat ρ of* **GHCoh** *using the canonical embedding $\mathcal{I}$: Dinat-***GHCoh** $\hookrightarrow$ *Dinat-***HCoh** *of Lemma 3.13.*

Proof. Note first that **HCoh** satisfies the three properties of Corollary 4.44: (i) *Dinat*-**HCoh** is soft (cf. Corollary 3.19) (ii) *Dinat*-**HCoh** is fully complete for MLL+Mix (cf. Proposition 3.7) (iii) `Mix` is monic in *Dinat*-**HCoh** (cf. under Proposition 2.6). Thus by Corollary 4.44 the result follows. □

Remark 4.56

1. In general, the class of proof-structures we obtain from dinaturals is a proper subset of all additive proof structures. The key point here is that those arising from legal total splittings automatically satisfy the no duplicate axiom-link property as well as the unique link property.
2. We also note that we have an algorithm (cf. Remark 4.17) for associating a proof-structure (eventually seen to be a net) with a **GHCoh** dinat. However, not all proof-nets are in the image of this construction. This arises for the same reason as Remark 4.12, namely the assignment of sequentializable MALL proof structures to proofs is not unique.

In what follows, for a dinat ρ, an arbitrarily fixed proof-structure Θ in $PS(\rho)$ is often denoted by Θ_ρ.

5 Simple Oriented Cycles in MALL Proof-Structures

We are interested in certain types of cycles which can arise in additive proof-structures. These cycles are called *simple oriented cycles*. *Orientedness* of cycles was first introduced in the work of Abramsky and Melliès [2, 3], which inspired our treatment here. However we introduce the notion of *simplicity* to further cut down the class of oriented cycles.

5.1 Simple Oriented Cycles

Our main results in this subsection are Lemma 5.2 and Lemma 5.8, which guarantee the existence of oriented cycles and of simple oriented cycles, respectively.

Definition 5.1 (oriented cycle) An *oriented cycle* is one in which the cycle has an orientation such that the induced direction on each proper jump goes from the conclusion of a &-link L to jump $S(L)$. See Figure 1 for the general shape of an oriented cycle, where an edge between a proper jump $S(L)$ and a conclusion of a &-link L is drawn with a dotted line.

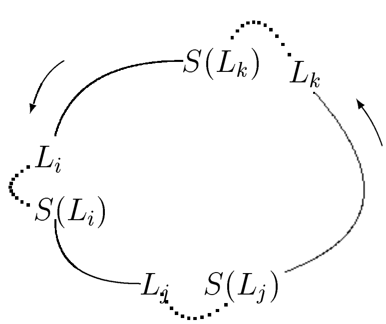


Figure 1: oriented cycle

Terminology:
Throughout this section, we say that a proof-structure Θ *has a cycle* if, for some switching S, the graph Θ_S has a cycle. A cycle $\mathtt{C}$ in Θ_S is often denoted by $(\mathtt{C}, S)$ so that a switch S yielding $\mathtt{C}$ is explicitly mentioned.

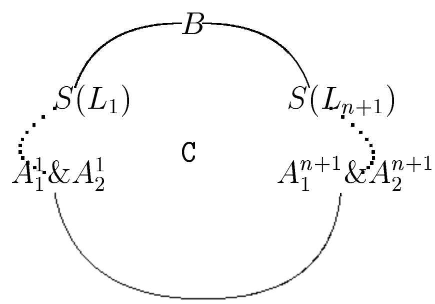
Lemma 5.2 (Transformation to oriented cycles) *Suppose Θ is a proof-structure such that Θ_{S_0} is connected for all normal switchings S_0. Every cycle $\mathtt{C}$ of Θ_S can be transformed into an oriented cycle $\check{\mathtt{C}}$ in $\Theta_{\check{S}}$ such that the valuation $\varphi_{\check{S}} = \varphi_S$. Hence, in particular, if Θ has a cycle, then it has an oriented cycle.*

Proof. It suffices to show that if a given cycle $(\mathtt{C}, S)$ is not oriented, then it can be transformed into a cycle $(\check{\mathtt{C}}, \check{S})$ satisfying the conditions in the lemma. Iterating this procedure yields the result. We shall prove this by induction on the number of proper jumps in a given $\mathtt{C}$.

Suppose an unoriented cycle $(\mathtt{C}, S)$ is given. Since every cycle which contains at most one proper jump can be oriented, we may assume that the number of proper jumps in $\mathtt{C}$ is $n+1$ with $n \geq 1$. We denote the list of all proper jumps by $S(L_1), \ldots, S(L_{n+1})$ in the order visited in the orientation of $\mathtt{C}$. We denote the conclusion of L_i by $A_1^i \& A_2^i$.

From the assumption of nonorientability of $\mathtt{C}$, we may assume without loss of generality that the induced directions on the proper jumps $S(L_1)$ and $S(L_{n+1})$ are different: i.e., $\mathtt{C}$ is of the following form, with B denoting a formula occurrence between $S(L_{n+1})$ and $S(L_1)$:

$$\mathtt{C} = B \cdots S(L_1)\ A_1^1 \& A_2^1 \cdots\ A_1^{n+1} \& A_2^{n+1}\ S(L_{n+1}) \cdots B$$



From the supposition, Θ_{S_0} is connected for the associated normal switching S_0 for S. Hence there is a path, say $\mathtt{p}$, between B and $A_1^{n+1} \& A_2^{n+1}$ in the graph Θ_{S_0}.

Let $A_1^k \& A_2^k$, $k \in \{1, \ldots n+1\}$, denote the designated &-formula on $\mathtt{C}$ which the path $\mathtt{p}$ (starting from B) first encounters. Then we may write $\mathtt{p}$ as

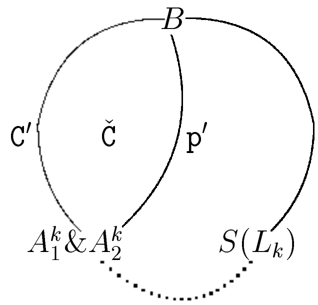
$$\mathtt{p} = \mathtt{p}'\ \mathtt{A}_1^{\mathtt{k}} \& \mathtt{A}_2^{\mathtt{k}}\ \mathtt{p}''.$$

where $\mathtt{p}''$ may be empty (when $k = n+1$).
On the other hand according to the two possible orientations for the jump $S(L_k)$, we may write $\mathtt{C}$ as one of the two following possibilities:

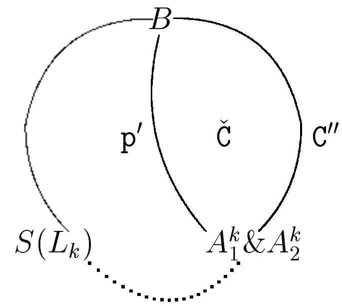
$$\mathtt{C} = \begin{cases} \mathtt{C}'\ A_1^k \& A_2^k\ S(L_k)\ \mathtt{C}'' & \text{(Case 1)} \\ \text{or} & \\ \mathtt{C}'\ S(L_k)\ A_1^k \& A_2^k\ \mathtt{C}'' & \text{(Case 2)} \end{cases}$$

In each case, we have a new cycle $\check{\mathtt{C}}$ (starting from B along with the path $\mathtt{p}'$ to $A_1^k \& A_2^k$ and ending at B) as follows:

(Case 1) $\check{\mathtt{C}} = \mathtt{p}'\ A_1^k \& A_2^k\ \mathtt{C}'$
(Case 2) $\check{\mathtt{C}} = \mathtt{p}'\ A_1^k \& A_2^k\ \mathtt{C}''$



Case 1



Case 2

In each case $\check{\mathtt{C}}$ has skipped the proper jump $S(L_k)$ (moreover every proper jump of $\check{\mathtt{C}}$ is one from $\mathtt{C}$). Hence the number of proper jumps of $\check{\mathtt{C}}$ is n, which is strictly less than that of $\mathtt{C}$, so the induction hypothesis applies. □

In addition to orientedness, we now introduce a *canonical shape* for cycles arising from proof-structures. Similar ideas are also developed in Abramsky and Melliès [2, 3].

Definition 5.3 (Canonical cycles) A cycle in a graph Θ_S is called *canonical* if the following two conditions are satisfied:

(i) Every proper jump on the cycle is to a conclusion of an axiom link.
(ii) Suppose A and B are formulas on the cycle. If A and B are nested in the subformula tree, then the orientation of the cycle induces a directed path from A to B or vice-versa. Suppose the path goes from A to B. Then that is the only directed path from A to B in the cycle.

Lemma 5.4 (Canonical cycles suffice) *For an arbitrary proof-structure Θ and a switching S, every cycle in Θ_S can be transformed into a canonical cycle in $\Theta_{S'}$, for some switching S' obtained from S.*

Proof. We prove (i) since (ii) is rather straightforward.

Given an arbitrary link K_i in $sl(\varphi_S(\Theta))$ whose conclusion is a proper jump $S(L_i)$, we have $w(K_i) \subset w(L_i)$ by the technical condition of Remark 4.6. Hereditarily above K_i in the slice $sl(\varphi_S(\Theta))$, there exists a link L_{i+1} (hence, $w(L_{i+1}) \subseteq w(K_i)$) which satisfies either of the following (a)$_i$ and (b)$_i$. In either case the graph Θ_S has subformula edges between $S(L_i)$ and a conclusion of L_{i+1}:

(a)$_i$: L_{i+1} is an axiom-link.
(b)$_i$: L_{i+1} is a &-link such that $S(L_{i+1})$ is a proper jump.

If L_{i+1} satisfies (a)$_i$, then we stop. If L_{i+1} does not satisfy (a)$_i$, hence satisfies (b)$_i$, then in $sl(\varphi_S(\Theta))$ we denote by K_{i+1} the conclusion of the proper jump $S(L_{i+1})$ guaranteed in (b)$_i$. Then by the same argument applied to K_{i+1}, there exists a link L_{i+2} hereditarily above K_{i+1}, which satisfies either of (a)$_{i+1}$ and (b)$_{i+1}$ and the graph Θ_S has subformula edges between $S(L_{i+1})$ and a conclusion of L_{i+2}. Thus in general, starting with $i = 1$, we have a series of links whose weights yield the following decreasing chain (strict inequalities come from the technical condition; non-strict inequalities from subformula relations):

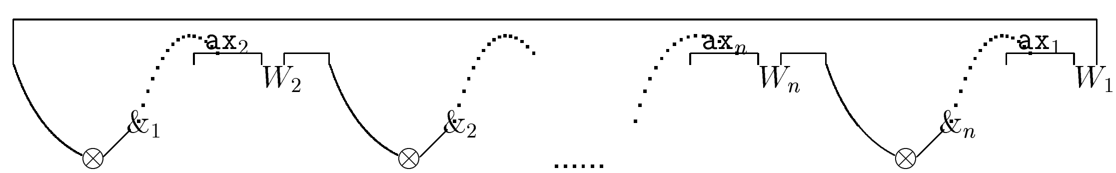
$$\cdots w(L_{i+1}) \subseteq w(K_i) \subset w(L_i) \subseteq \cdots \subset w(L_2) \subseteq w(K_1) \subset w(L_1)$$

Note that the chain stops if L_{i+1} satisfies (a)$_i$. If p_i denotes an eigenweight for the &-link L_i, then $w(K_i)$ depends on p_i $(i = 1, \dots)$. For example, it will turn out that in Figure 2 below, if $L_1 = \&_1$ then $L_{i+1} = \mathtt{ax}_2$.

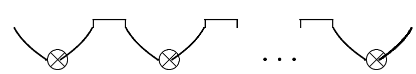
Now we claim that there exists i such that L_{i+1} satisfies (a)$_i$: *Intuitively, this means that jumps to axiom links suffice.* For the proof, suppose otherwise. Then by virtue of the fact that the number of &-links in Θ is finite, L_{i+1} becomes identical to a previous L_j $(j < i+1)$, hence from the above chain, we have $w(L_j) = w(L_{i+1}) \subseteq w(K_i) \subset \cdots \subset w(L_{j+1}) \subseteq w(K_j) \subset w(L_j) = w(L_{i+1})$. This is a contradiction since it implies $w(L_{i+1}) \subset w(L_{i+1})$.

Now we show that cycles remain when one jumps to axiom links. Given an arbitrary proper jump $S(L_1)$ lying on a cycle, we may change the switching S into S' by defining $S'(L_1)$ to be a conclusion of the axiom link L_{i+1} guaranteed in the above paragraph. This choice of jump is possible because $w(L_{i+1})$ depends on p_1: this arises from Lemma 4.5 together with the fact that $w(L_{i+1}) \subseteq w(K_1)$. It is straightforward that a cycle still occurs in $\Theta_{S'}$. □

Thus from now on we always consider canonical cycles. In particular the general shape of an oriented canonical cycle is shown in Figure 2. We draw the proper jumps to axiom links to make the picture more readable. Note that the shape of this oriented cycle implies that $w(\mathtt{ax}_{i+1})$ depends on p_i, for each $i = 1, \dots, n$.



where each W_i is a graph of the form



or empty (in the latter case the two axiom links attached to W_i are identified).

Figure 2: oriented canonical cycle

Remark 5.5 In a canonical oriented cycle of the form in Figure 2, we may assume that the left-conclusion of $\mathtt{ax}_{i+1}$ cannot be a subformula of the $\&_i$-formula, because if it were, there would be a series of subformula connections between the $\&_i$-formula and the left-conclusion of $\mathtt{ax}_{i+1}$, so there would be no need for a proper jump.

The main contribution of this subsection is to further cut down the class of cycles arising in a connected proof-structure. The cycles we consider are called *simple* cycles:

Definition 5.6 (simple cycle) A cycle $\mathtt{C}$ in a graph Θ_S is called *simple* if the following holds for every link K in $sl(\varphi_S(\Theta))$ whose conclusion is a proper jump $S(L)$ lying on the cycle $\mathtt{C}$:

$w(K) = \epsilon p_L.v$	where $\epsilon \in \{1, \neg\}$, p_L is the associated eigenweight for the &-link L and v does not depend on any eigenweights associated with &-links whose conclusions lie on $\mathtt{C}$.

In particular when a given cycle is oriented and canonical as in Figure 2, it is *simple* if for all $i = 1, \ldots, n$ the following holds:

$w(\mathtt{ax}_{i+1}) = \epsilon_i p_i.v_i \pmod n$	where $\epsilon_i \in \{1, \neg\}$ and v_i does not depend on any eigenweight p_j with $1 \leq j \leq n$.

The following is an important property of simple cycles:

Lemma 5.7 (Weight lemma for simple cycles)
For a simple cycle $\mathtt{C}$ in Θ_S, let $L_1, \ldots, L_n$ denote the list of all &-links in $sl(\varphi_S(\Theta))$ whose conclusions lie on $\mathtt{C}$. Then for every i, $w(L_i)$ does not depend on any p_j with $1 \leq j \leq n$.

In particular when a given simple cycle is an oriented canonical cycle as in Figure 2, the following holds: For $i \leq n$, $w(L_i)$ does not depend on any p_j ($1 \leq j \leq n$) where L_i is the &-link whose conclusion is $-\&_i-$, the ith distinguished &-formula in $sl(\varphi_S(\Theta))$.

Proof. First of all, we recall the technical condition for proof-structures (cf. Definition 4.3 and Remark 4.6). If a weight v in Θ depends on p_i then $v \subseteq w(L_i)$. Now suppose that $w(L_i)$ depends on p_j. If $i = j$, this contradicts the condition that $w(L_i)$ does not depend on p_i. If $i \neq j$, then the technical condition for L_i together with Lemma 4.5 says the following: (v depends on p_i) implies (v depends on both p_i and p_j). When applied to $v = w(\mathtt{ax}_{i+1})$, this contradicts the simplicity of $\mathtt{C}$ (which implies that $w(\mathtt{ax}_{i+1})$ does not depend on p_j). Hence we have the conclusion of the Lemma. □

The following is an important lemma for obtaining simple cycles from oriented ones.

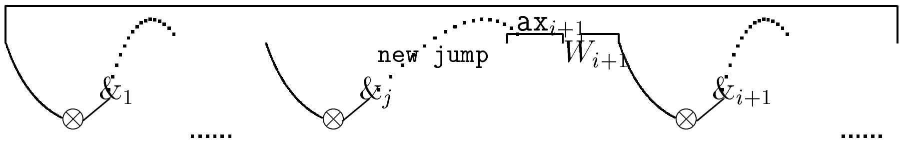
Lemma 5.8 (Transformation to simple oriented cycles) *Every oriented cycle $\mathtt{D}$ of Θ_S can be transformed into a simple oriented cycle $\mathtt{D}'$ of $\Theta_{S'}$ such that $\varphi_{S'} = \varphi_S$. Hence, in particular, if an arbitrary proof-structure Θ has an oriented cycle then it has a simple oriented cycle.*

Proof. We show that if a given oriented cycle $(\mathtt{D}, S)$ is not simple, then it can be transformed into an oriented cycle $(\mathtt{D}', S')$ satisfying the conditions of the Lemma.

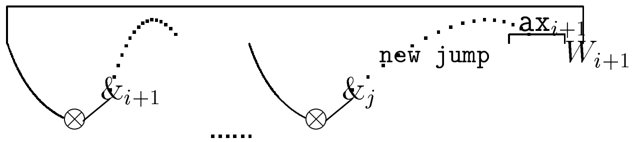
In our proof of this lemma, Girard's technical condition for proof-structures is critical.

We may assume that the given $\mathtt{D}$ is of the form in Figure 2. We know that $w(\mathtt{ax}_{i+1})$ depends on p_i for all i, since a conclusion of $\mathtt{ax}_{i+1}$ is a jump for $\&_i$. Suppose $\mathtt{D}$ is not simple; i.e., there exists i such that $w(\mathtt{ax}_{i+1})$ depends on p_j with $j \neq i$. Then a new jump edge can be drawn between $\&_j$ and $\mathtt{ax}_{i+1}$, which results in another oriented canonical cycle whose number of jump edges is strictly smaller than n. See the figures below for a new jump edge together with the resulting oriented canonical cycle for each case depending on whether $j < i$ or $j > i$.

(The case $j < i$):



(The case $j > i$):



As is clear from the figures, in either case the resulting oriented canonical cycle has a smaller number of jumps and every jump of the new cycle is one from the original cycle. □

5.2 Global Simple Oriented Cycle

We say that a cycle C *passes through a link* L if the conclusions of L lie on C.

Definition 5.9 (global cycle) A cycle C in a proof-structure Θ is *global* if C passes through all &-links whose weights are 1 in Θ.

In the following, for an eigenweight r, $\&_r$ denotes the associated &-link.

Lemma 5.10 (weight lemma for a global simple oriented cycle) *For a simple oriented cycle* C*, if* C *is global, then the following hold:*

(i) $w(L) = 1$ *for the* L *of Definition 5.6, hence* $w(L_i) = 1$ *for the* L_i *in Lemma 5.7. That is, all &-links which cause proper jumps have weight* 1*.*

(ii) For the weight $w(\mathtt{ax}_{i+1}) = \epsilon p_i.v_i$ *of Definition 5.6, if the* v_i *depends on an eigenweight* r*, then* $w(\&_r)$ *depends on the eigenweight* p_i*.*

(iii) For the weights $w(\mathtt{ax}_{i+1}) = \epsilon p_i.v_i$ *and* $w(\mathtt{ax}_{j+1}) = \epsilon p_i.v_j$ *of Definition 5.6, if* $i \neq j$ *then the eigenweights on which* v_i *depends are disjoint from those on which* v_j *depends.*

Proof. (i) Suppose for contradiction that $w(L) \neq 1$; i.e., that $w(L)$ depends on some eigenweight, say r_1. We obtain the contradiction using an inductively defined series of steps. As step 1, we have following:

$$w(L) \subset w(\&_{r_1}) \qquad \cdots (1.1)$$

$$w(\&_{r_1}) \neq 1\text{; i.e., } w(\&_{r_1}) \text{ depends on some eigenweight, say } r_2. \qquad \cdots (1.2)$$

Condition 1.1 is Girard's technical condition (cf. Remark 4.6). Condition 1.2 is obtained as follows. First, Lemma 5.7 implies that the conclusion of the &-link $\&_{r_1}$ does not lie on C. Second, since C is global, we conclude 1.2.

Step 1 induces step 2:

$$w(L) \subset w(\&_{r_1}) \subset w(\&_{r_2}) \qquad \cdots (2.1)$$

$$w(\&_{r_2}) \neq 1\text{; i.e., } w(\&_{r_2}) \text{ depends on some eigenweight, say } r_3. \qquad \cdots (2.2)$$

2.1 is from 1.1 and Girard's technical condition. 2.2 is obtained as follows. First, Lemma 4.5 together 1.2 and 2.1 says that $w(L)$ depends on r_2. Second, Lemma 5.7 says that the conclusion of the &-link $\&_{r_2}$ does not lie on C. Third, since C is global, we conclude 2.2.

Step 2 induces step 3, and in general we have step n, which gives rise to the following strictly increasing *infinite* sequence of weights:

$$w(L) \subset w(\&_{r_1}) \subset w(\&_{r_2}) \subset \cdots \subset w(\&_{r_n}) \subset \cdots$$

Since the number of &-links in Θ is finite, this is impossible, hence we have a contradiction.

(ii) On the one hand from Girard's technical condition, we have $w(\mathtt{ax}_{i+1}) := \epsilon p_i.v_i \subset w(\&_r)$. On the other hand Lemma 5.7 says that the conclusion of the &-link $\&_{r_1}$ does not lie on the cycle $\mathtt{C}$. Since $\mathtt{C}$ is global, we have $w(\&_r) \neq 1$; i.e., $w(\&_r)$ depends on some eigenweight, say r_1. Then from Girard's technical condition, we have $w(\&_r) \subset w(\&_{r_1})$. If $w(\&_{r_1}) = 1$, we stop. Otherwise $w(\&_{r_1})$ depends on some eigenweight, say $\&_2$. Then $w(\&_{r_1}) \subset w(\&_{r_2})$ from Girard's technical condition. By repeating this, we have a sequence

$$w(\mathtt{ax}_{i+1}) := \epsilon p_i.v_i \subset w(\&_r) \subset w(\&_{r_1}) \subset \cdots \subset w(\&_{r_n+1}) \subset \cdots$$

such that $w(\&_{r_m})$ depends on eigenweight r_{m+1} for each m.

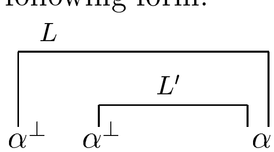
It is important to observe that the sequence terminates; i.e., $w(\&_{r_n+1}) = 1$ for some $n \geq 0$. This is because the boolean algebra of weights is finitely generated. Since $\mathtt{C}$ is global, r_{n+1} must be p_k for some k. Since $w(\&_{r_n})$ depends on r_{n+1}, which is p_k, the above sequence together with Lemma 4.5 implies that both $w(\&_r)$ and $w(\mathtt{ax}_{i+1})$ depend on p_k. From the definition of simple cycle, the only possible p_k on which $w(\mathtt{ax}_{i+1})$ depends is p_i; i.e., $i = k$. Thus we have derived the assertion.

(iii) This is a direct corollary of (ii): Suppose for contradiction that there exists a common eigenweight r_1 on which both v_i and v_j depend. On the one hand, by applying (ii) to v_i, we know that $w(\&_r)$ depends on p_i. On the other hand by applying Girard's technical condition to $w(\mathtt{ax}_{j+1})$, we have $w(\mathtt{ax}_{j+1}) := \epsilon p_j.v_j \subset w(\&_r)$. These imply with Lemma 4.5 that $w(\mathtt{ax}_{j+1}) := \epsilon p_j.v_j$ depends on p_i. From the definition of simple cycle $\mathtt{C}$ (cf. Definition 5.6), the only weight among $p_1, \ldots, p_n$ on which $w(\mathtt{ax}_{j+1})$ depends is p_j. Thus we have a contradiction, since $i \neq j$. □

We now introduce a fundamental property in this subsection. We shall be interested in proof-structures with the no duplicate axiom-link (NDAL) property (cf. Definition 4.10).

Definition 5.11 (A valuation yields two distinct axiom-links) Let Θ be an NDAL proof-structure and α a literal in Θ. We say a valuation φ yields *two distinct axiom-links* w.r.t an eigenweight p and a literal α if the following holds:

- The axiom links L in $sl(\varphi(\Theta))$ and L' in $sl(\varphi'(\Theta))$ with conclusion α have different conclusions, where φ' is the same as φ but $\varphi'(p) = \neg\varphi(p)$. (We note that in a slice, there is a unique link whose conclusion is a fixed literal. That is, L is the unique link with conclusion α in $sl(\varphi(\Theta))$ and L' is that in $sl(\varphi'(\Theta))$. Thus in Θ the two axiom links have the following form:



Remark 5.12 Note that the weights of the two axiom-links L and L' in the above Definition 5.11 depend on p.

Next, we prove the following lemma, for which the above Lemma 5.10 (ii) is crucial.

Lemma 5.13 (Existence of two distinct axiom-links) *Suppose Θ is an NDAL proof-structure and $\mathtt{C}$ is a global simple oriented cycle in Θ. For the weight $w(\mathtt{ax}_{i+1}) := \epsilon p_i.v_i$ in Definition 5.6, let $\{r_1, \dots, r_m\}$ denote the set of eigenweights on which v_i depends, and let α_{i+1} denote $\mathtt{ax_{i+1}}$'s conclusion lying on the cycle $\mathtt{C}$. If $w(\alpha_{i+1}) = 1$, then there exists a valuation ψ_i for $\{r_1, \dots, r_m\}$ such that every one of its extensions $\bar{\psi_i}$ to a valuation for Θ yields two distinct axiom-links with respect to p_i and α_{i+1}.*

Proof. We may suppose without loss of generality that $\epsilon = 1$; i.e., $w(\mathtt{ax}_{i+1}) = p_i.v_i$. We shall define a valuation ψ_i by induction on m.

(Base Case) The case where $m = 0$:

In this case $w(\mathtt{ax}_{i+1}) = p_i$. Using the no duplicate axiom-link property of Θ, observe: since $w(\alpha_{i+1}) = 1$, there must exist an axiom link $\overline{\mathtt{ax}}$ one of whose conclusions is α_{i+1}, but whose other conclusion is a different occurrence from that of $\mathtt{ax}_{i+1}$. Thus the assertion is straightforward.

(Induction Case) The case where $m \geq 1$:

Consider a set $\mathbf{A}$ of axiom-links $\overline{\mathtt{ax}}$ one of whose conclusions is α_{i+1} and whose other conclusion is different from that of $\mathtt{ax}_{i+1}$. The no duplicate axiom-link property guarantees that the set $\mathbf{A}$ is nonempty. If no weight of $\overline{\mathtt{ax}}$ in $\mathbf{A}$ depends on any r_k, then the assertion is straightforward as in the Base Case, because in this case $w(\overline{\mathtt{ax}}) = \neg p_i$ since the cycle is simple. Thus in the following we may assume that *there exists an $\overline{\mathtt{ax}}$ in $\mathbf{A}$ such that $w(\overline{\mathtt{ax}})$ depends on some r_k, which in the following will be simply denoted by r.*

(Case 1) The case where the occurrence $\mathtt{ax}_{i+1}$ remains after setting $r = 1$:

In this case, the valuation $r = 1$ preserves the no duplicate axiom-link property of conclusion α_{i+1}. Thus the assertion is reduced to the induction hypothesis by defining $\psi_i(r) = 1$.

(Case 2) The case where $\mathtt{ax}_{i+1}$ disappears after setting $r = 1$:

In this case, $w(\mathtt{ax}_{i+1})$ has an occurrence of $\neg r$ along with v_i. Thus from Girard's technical condition, $p_i.v_i := w(\mathtt{ax}_{i+1}) \subset w(\&_r)$. Thus, from Lemma 5.10 (ii) , $w(\&_r)$ depends on p_i , so $w(\&_r)$ must have an occurrence of p_i. Hence we note the following important fact:

$$\text{Every weight } w \text{ depending on } r \text{ has an occurrence of } p_i \tag{2}$$

(2) is obtained as follows: First, from Girard's technical condition $w \subset w(\&_r)$. Second, since p_i occurs in $w(\&_r)$, it occurs in w as well, by Lemma 4.5.

(Case 2.1) The case where the valuation $r = 1$ gives rise to the duplicate axiom-links of the form in Definition 4.10 one of whose conclusions is α_{i+1}.

Since $w(\alpha_{i+1})$, which is 1, has no occurrence of p_i and is a disjoint sum of weights of axiom-links with conclusion α_{i+1}, (2) implies that there exists an axiom-link $\underline{\mathtt{ax}}$ with conclusion α_{i+1} such that $w(\underline{\mathtt{ax}})$ does not depend on r; i.e., $\underline{\mathtt{ax}}$ remains under both valuations $r = 1$ and $r = 0$.

(Case 2.1.1) The case where the two conclusions of $\underline{\mathtt{ax}}$ coincide with those of $\mathtt{ax}_{i+1}$:

This case guarantees that the duplicate axiom-links after the valuation $r = 1$ share two

conclusions of $\underline{\mathtt{ax}}$. Thus $\overline{\mathtt{ax}}$ disappears after the valuation $r = 1$, hence $w(\overline{\mathtt{ax}})$ must have an occurrence of $\neg r$. Now let us set $r = 0$, which retains both occurrences of $\underline{\mathtt{ax}}$ and $\overline{\mathtt{ax}}$; thus in this case, after the valuation, the no duplicate axiom-link property of conclusion α_{i+1} still holds. Thus the assertion is reduced to the induction hypothesis by defining $\psi_i(r) = 0$.
(Case 2.1.2) The negation of Case 2.1.1:
In this case a conclusion other than α_{i+1} of $\underline{\mathtt{ax}}$ differs from that of $\mathtt{ax}_{i+1}$. Now let us set $r = 0$, a valuation which retains both occurrences $\mathtt{ax}_{i+1}$ and $\underline{\mathtt{ax}}$. Hence under the valuation, the no duplicate axiom-link property of conclusion α_{i+1} is preserved. Thus the assertion is reduced to the induction hypothesis by defining $\psi_i(r) = 0$.
(Case 2.2): The negation of Case 2.1:
This case guarantees that the no duplicate axiom-link property of conclusion α_{i+1} holds under the valuation $r = 1$. Thus the assertion directly reduces to the induction hypothesis by defining $\psi_i(r) = 1$. □

Now we are ready to state the goal of this subsection.

Corollary 5.14 (Existence of two distinct axiom-links in global cycles) *Suppose a proof-structure* Θ *has a global simple oriented cycle* $\mathtt{C}$ *such that* $w(\alpha_{i+1}) = 1$ *for all* $i \in \{1, \ldots, n\}$. *Then there exists a switching* S *such that* $\mathtt{C}$ *appears in* Θ_S *and its valuation* φ_S *yields two distinct axiom-links with respect to* p_i *and* α_{i+1} *for all* $i \in \{1, \ldots, n\}$.

Proof. Since $w(\alpha_{i+1}) = 1$, Lemma 5.13 guarantees, for each i, the existence of a valuation ψ_i any of whose extensions to a whole valuation for Θ yields two distinct axiom-links with respect to p_i and α_{i+1}. On the other hand, by virtue of Lemma 5.10 (iii), the valuation ψ_i for each i is *simultaneously* extendable to a valuation ψ for Θ, thus ψ yields two distinct axiom-links with respect to p_i and α_{i+1} for all i. Since $w(L_i) = 1$ from Lemma 5.10 (i) and $w(\alpha_{i+1}) = 1$ from the assumption above, all edges constituting $\mathtt{C}$ except proper jumps are retained under an arbitrary valuation, hence under ψ. Moreover by Remark 5.12, we can draw a jump from L_i to α_{i+1} for all i in $sl(\psi(\Theta))$. A switching S is defined from the valuation ψ together with these choices of jumps so that the cycle $\mathtt{C}$ is retained in Θ_S. □

5.3 On Cycles and Connectedness of MALL Proof-Structures

In this last subsection we present various geometrical properties of MALL proof structures. In the Main Proposition 5.15 we characterize MALL proof nets among certain connected MALL proof structures. This is a direct corollary of Lemmas 5.2, 5.4, and 5.8. Second, we derive two lemmas (on connectedness and oriented cycles) specifically for proof-structures arising from dinats. These lemmas are used in proving our main Full Completeness Theorem (Theorem 6.2 in Section 6 below).

Proposition 5.15 (Main Proposition on Simple Oriented Cycles) *Let* Θ *be an arbitrary* MALL *proof-structure. If* Θ_S *is connected for all normal switchings* S, *either* (*i*) *or* (*ii*) *holds:*

(i) Θ *is a proof-net.*
(ii) Θ *has a simple oriented cycle.*

Proof. Suppose Θ is not a proof-net. We show Θ has a simple oriented cycle. From the connectedness of Θ_S, we know there must be a cycle in Θ, for some switch setting. From

Lemma 5.2 that cycle can be transformed into an oriented cycle. From Lemma 5.8, the oriented cycle can be transformed into a simple oriented cycle. □

The following lemma is the crucial place where we make use of the double gluing construction, applied to the category **HCoh**. As in the work of Tan [34], application of double gluing yields a model which does not validate the Mix rule, and in this case is fully complete for MLL. This lemma also illustrates the key point: working in **GHCoh** forces the associated proof-structures to be connected.

Lemma 5.16 (connectedness of Θ_ρ under normal switchings) *For an arbitrary ρ in Dinat-**GHCoh** , $(\Theta_\rho)_S$ is connected for every normal switching S.*

Proof. First observe that by definition every switching S uniquely determines a valuation φ_S on eigenweights. Hence this valuation yields a slice $sl(\varphi_S(\Theta))$ which we identify with an MLL proof-structure (cf. Remark 4.19). Moreover if Θ_ρ is a proof-structure associated with a dinat ρ, then for an arbitrary switching S, there is a dinat $\varphi_S(\rho)$ of MLL type such that

$$sl(\varphi_S(\Theta_\rho)) = \Theta_{\varphi_S(\rho)} \tag{3}$$

Second, for every MLL proof-structure of the form $sl(\varphi_S(\Theta))$ the graph $(sl(\varphi_S(\Theta)))_S$ is drawn as usual by the choice of ⅋-switchings determined by S; and we have the following for an arbitrary normal switching S:

$$\text{The graph } \Theta_S \text{ is connected iff the graph } (sl(\varphi_S(\Theta)))_S \text{ is connected} \tag{4}$$

The MLL full completeness of *Dinat*-**GHCoh** (Proposition 3.17) implies that $\Theta_{\varphi_S(\rho)}$ is an MLL proof-net, hence in particular $(sl(\varphi_S(\Theta_\rho)))_S$ is connected. Thus the assertion follows from the above observations (3) and (4). □

For the final result in this subsection, we prove the following lemma, which is the main consequence of the Fundamental Proposition (Proposition 4.47) in Section 4.5. The lemma will be used in the proof of the Main Theorem (Theorem 6.2) in the next Section 6. Before reading this lemma, the reader should examine the Appendix to this section (subsection 5.4), which gives the background on &-semi-simple types.

Lemma 5.17 (Existence of global cycles in associated proof structures) *Consider the set $\mathcal{S}$ of **HCoh** -dinats ρ of &-semi-simple type (in the sense of subsection 5.4) such that there is a Θ in $PS(\rho)$ and Θ has a cycle. If the set $\mathcal{S}$ is non empty, then there exists a pair (ρ, Θ) consisting of a dinat $\rho \in \mathcal{S}$ and a proof structure $\Theta \in PS(\rho)$ such that every cycle in Θ is global.*

Proof. Take a minimal dinat $\rho \in \mathcal{S}$ w.r.t the lexicographic ordering on the following pairs:

$$(\text{number of } \otimes\text{'s in } \rho\text{'s type, number of } \{⅋, \&, \oplus\}\text{'s in } \rho\text{'s type})$$

From Corollary 4.50 of the Fundamental Proposition, together with the minimality of the size, ρ cannot be further semantically split; i.e., the type of ρ has no outermost $\{\&, ⅋\}$ and has no outermost $\{\otimes, \oplus\}$ which can be semantically split. Moreover ρ is not the union of two dinats via the Mix-rule. By the Fundamental Proposition 4.47, the proof-structure counterpart to this is the following:

$\forall\,\Theta \in PS(\rho)$, Θ has no terminal $\otimes$-link which can be split and no terminal $\{\&, \parr, \oplus_1, \oplus_2\}$-links. Moreover Θ is not the union of two proof-structures. (#)

We begin by proving the following:

(Claim): for a ρ as above, for every $\Theta \in PS(\rho)$ and for every &-link L of weight 1 in Θ, there exists a $\otimes$-link L' *immediately* below L.

First, we shall show that there exists a $\otimes$-link hereditarily below L. Suppose for contradiction that this is false. Since $w(L) = 1$, there cannot exist any other &-link hereditarily below L. Thus either L is terminal or all links hereditarily below L are $\{\parr, \oplus_1, \oplus_2\}$-links, whose weights are 1. This means that Θ must have a terminal $\{\&, \parr, \oplus_1, \oplus_2\}$-link, which contradicts (#). Thus there exists a $\otimes$-link hereditarily below L.

Now consider the uppermost $\otimes$-link, say L', hereditarily below the &-link L. We shall show that this is the L' of the claim, i.e., L' is *immediately* below L. We first observe that there can be no $\{\oplus_1, \oplus_2\}$-link hereditarily below L. For suppose otherwise. Then such a $\{\oplus_1, \oplus_2\}$-link would have weight 1, which corresponds to a semantically redundant $\oplus$-connective of ρ. This would contradict the minimality of the size of ρ. So the link immediately below L must be a $\{\parr, \otimes\}$-link. When it is $\parr$, there exists a $\parr$-link immediately above the $\otimes$-link L'. But this contradicts the semi-simplicity of ρ, since a linear distributivitivity of subsection 5.4 can be applied. Thus we conclude that the link immediately below L must be a $\otimes$-link, which proves the Claim.

Note that since $\rho \in \mathcal{S}$, there exists $\Theta \in PS(\rho)$ such that Θ has a cycle. We shall show this pair (ρ, Θ) is the one asserted in the Lemma. Suppose for contradiction that Θ has a non-global cycle; i.e., there exists a cycle $\mathtt{C}$ in Θ and there exists a &-link L of weight 1 such that $\mathtt{C}$ does not pass through L. From the above Claim, there exists a $\otimes$-link L' immediately below L. From ρ we apply a Mix to the $\otimes$ corresponding to L', to obtain a **HCoh**-dinat ρ'; i.e.,

$$\rho' := \rho[A \parr (B\&C)] \text{ where } \rho = \rho[A \otimes (B\&C)]$$

In the above, $B\&C$ is the conclusion of L, hence $A \otimes (B\&C)$ is the conclusion of L'. Now a proof-structure Θ' is defined to be one obtained from Θ by replacing the $\otimes$-link L (together with hereditary occurrences of $\otimes$) by a $\parr$-link (together with occurrences of $\parr$). Then we have $\Theta' \in PS(\rho')$. It is important to observe that, since the simple oriented cycle $\mathtt{C}$ does not pass through L, the cycle $\mathtt{C}$ is retained in Θ'. Thus it holds that $\rho' \in \mathcal{S}$. Note that the size of ρ' is strictly smaller than that of ρ; i.e., in the above lexicographic ordering, the level of ρ' is strictly lower than that of ρ.

By means of reductions to &-semi-simple sequents, (ρ', Θ') can be reduced to a certain pair (ρ'', Θ'') such that ρ'' is a dinat of &-semi-simple type and the simple oriented cycle $\mathtt{C}$ is retained in $\Theta'' \in PS(\rho'')$. Thus we have that $\rho'' \in S$. Since the size of ρ'' is strictly smaller than that of ρ, this contradicts the minimality of the size of ρ.

□

5.4 Appendix: reduction to &-Semi-Simple Sequents

In this subsection we introduce some syntactical notions. These are used in Lemma 5.17 of Subsection 5.3 above and in Section 6 below. We consider MALL formulas as being generated from literals using the connectives $\otimes, \parr, \&, \oplus$, but no units.

Definition 5.18 A *covariant context* (context, for short) is a sequent generated from distinguished constant symbols called *holes* together with literals using the MALL connectives and in which any holes occur exactly once. We denote a context Γ with distinguished holes $*_1, \ldots, *_n$ by $\Gamma[*_1, \ldots, *_n]$. We may substitute arbitrary formulas for holes in a context: we write $\Gamma[A_1, \ldots, A_n]$ for the context $\Gamma[*_1, \ldots, *_n]$ with $*_i$ replaced by A_i. A hole $*$ has a *multiplicative occurrence* in context Γ if in the parsing tree of the context, all connectives on the unique path from $*$ to the root are multiplicatives.

Example 5.19 In the context $\Gamma[*] = (* \otimes (X \oplus Y)) \parr (Z \& W)$, $*$ occurs multiplicatively, whereas in the contexts $\Gamma_1[*] = (* \otimes (X \oplus Y)) \& (Z \oplus W)$ and $\Gamma_2[*] = (* \oplus (X \oplus Y)) \parr (Z \oplus W)$, $*$ does not occur multiplicatively.

We define M$\oplus$LL analogously to $\parr$ALL: it is the fragment of MALL generated using just the MLL and $\oplus$ connectives. We now extend the notion of *semi-simple sequent* as in Hyland-Ong [29] to M$\oplus$LL:

Definition 5.20 (M$\oplus$LL semi-simple sequent) An M$\oplus$LL sequent Γ is *semi-simple* if it has the form $\Gamma[\ell_{1,1} \otimes \ell_{1,2} \otimes \cdots \otimes \ell_{1,m_1}, \ldots, \ell_{n,1} \otimes \ell_{n,2} \otimes \cdots \otimes \ell_{n,m_n}]$, where $\Gamma[*_1, \ldots, *_n]$ is a context constructed using *only the connectives* $\parr, \oplus$ and the ℓ_{ij} are literals.

We now introduce the analog of the theorem in [29] which shows it suffices to consider semi-simple sequents in proofs of Full Completeness:

Proposition 5.21 (Reduction to semi-simple sequents) *Suppose* $\vdash \Gamma$ *is an* M$\oplus$LL *sequent. Then there exists a list of* M$\oplus$LL *semi-simple sequents* $\vdash \Gamma_1, \vdash \Gamma_2, \cdots, \vdash \Gamma_n$ *such that* $\vdash \Gamma$ *is provable iff for all i,* $\vdash \Gamma_i$ *is provable (in* M$\oplus$LL *).*

The proof is similar to Hyland-Ong [29]. First we need 3 preliminary syntactic lemmas. In each case, it suffices to state them for contexts with one hole.

Lemma 5.22 *Let* $\Gamma = \Gamma[A \otimes (B \parr C)]$ *be a* MALL *-sequent. Let* $\Gamma_1 = \Gamma[(A \otimes B) \parr C]$ *and* $\Gamma_2 = \Gamma[(A \otimes C) \parr B]$*. Then we have:*

(i) For all $i = 1, 2 \quad \vdash \Gamma \multimap \Gamma_i$ *is provable.*

(ii) $\vdash \Gamma$ *is provable if and only if* $\vdash \Gamma_i$ *is provable for* $i = 1, 2$.

Lemma 5.23 *Let* $\Gamma = \Gamma[A \otimes (B \otimes C)]$ *be a* MALL *-sequent. Let* $\Gamma_1 = \Gamma[A \otimes (B \parr C)]$ *and* $\Gamma_2 = \Gamma[A \parr (B \otimes C)]$*. Then we have:*

(i) For all $i = 1, 2 \quad \vdash \Gamma \multimap \Gamma_i$ *is provable.*

(ii) $\vdash \Gamma$ *is provable if and only if* $\vdash \Gamma_i$ *is provable for* $i = 1, 2$.

Proof of Lemmas 5.22 and 5.23: The proofs are the same as in Hyland-Ong [29], observing that the "if" direction of part (ii) of each Lemma is still valid using MALL proof-nets, not just ones for MLL . □

Finally, let Γ be an M$\oplus$LL sequent, as above.

Lemma 5.24 $\Gamma[A \otimes (B \oplus C)]$ *is provable iff* $\Gamma[(A \otimes B) \oplus (A \otimes C)]$ *is provable.*

Proof. We can prove $A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)$ and $(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)$ □

Proof of Proposition 5.21 Suppose Γ is an M$\oplus$LL sequent. Since $\otimes$ distributes over ⅋ and $\oplus$ by the Lemmas, we use this fact to push occurrences of $\otimes$ inward. We obtain sequents of the form $\Gamma[\ell_{1,1} \otimes \ell_{1,2} \otimes \cdots \otimes \ell_{1,m_1}, \ldots, \ell_{n,1} \otimes \ell_{n,2} \otimes \cdots \otimes \ell_{n,m_n}]$. □

On a semantic level, every $*$-autonomous category with products and coproducts has the following natural morphisms (which are monic in the case of **Coh** and **HCoh** , hence in particular **GHCoh**). These correspond to the sequents in the above syntactic lemmas.

1. *Linear Distributivities*:
   (a) $A \otimes (B ⅋ C) \to (A \otimes B) ⅋ C$
   (b) $A \otimes (B ⅋ C) \to (A \otimes C) ⅋ B$
2. *Distribution of $\otimes$ over $\oplus$* : $A \otimes (B \oplus C) \xrightarrow{\simeq} (A \otimes B) \oplus (A \otimes C)$

The above morphisms are actually *natural transformations*, thus compose with dinats [9]. Hence, as in Proposition 5.21 any M$\oplus$LL dinat $\rho : \mathbf{1} \to \Gamma$ yields (by composition) a list of dinats $\{\rho_i : \mathbf{1} \to \Gamma_i \mid 1 \leq i \leq n\}$ where the Γ_i are semi-simple sequents.

Definition 5.25 (&-semi-simple MALL sequent) A MALL sequent Γ is called *&-semi-simple* if it is of the form $\Gamma[A_{1,1}\&A_{1,2}, \cdots, A_{n,1}\&A_{n,2}]$ where $\Gamma[*_1, \cdots, *_n]$ is an M$\oplus$LL *semi-simple context*, i.e. a context in which, if we replace the holes by literals, we obtain a semi-simple M$\oplus$LL sequent. Here the A_{ij} may be arbitrary MALL formulas.

In other words, Γ is &-semi-simple if, whenever we replace the outermost occurrences of &–together with the scoping formulas–by holes, then the resulting context is M$\oplus$LL semi-simple.

Example 5.26

1. $\vdash ((A\&B) \otimes \ell \otimes r) \oplus C$ *is &-semi-simple , where A, B are* MALL *formulas, l and r are literals, and C is a* {⅋ , $\oplus$}*-formula.*
2. $\vdash p^\perp \oplus q, (p\&q^\perp) \otimes r^\perp \otimes ((s ⅋ s^\perp)\&(t ⅋ t^\perp)), r$ *is &-semi-simple ,* MALL*-provable sequent where p, q, r, s and t are atoms.*

The proof of Proposition 5.21 in fact applies to &-semi-simple sequents verbatim, i.e.

Proposition 5.27 (Reduction to &-semi-simple Sequents) *Suppose* $\vdash \Gamma$ *is a* MALL *sequent. Then there exists a list of* MALL *&-semi-simple sequents* $\vdash \Gamma_1, \vdash \Gamma_2, \cdots, \vdash \Gamma_n$ *such that* $\vdash \Gamma$ *is provable iff for all i,* $\vdash \Gamma_i$ *is provable (in* MALL*).*

6 MALL Full Completeness in GHCoh

Our purpose in this section is to prove MALL full completeness in **GHCoh** (Theorem 6.4). Namely, we shall show that the proof-structure Θ_ρ associated to a dinatural ρ in Corollary 4.55 is a proof-net.

According to Corollary 4.55, we are interested in proof structures $\Theta = \Theta_\rho$ arising from dinats $\rho : \mathbf{1_G} \to \Delta$ of **GHCoh**. Given Proposition 5.27 of Subsection 5.4, from now on we only consider dinats $\rho : \mathbf{1_G} \to \Delta$ whose type is a &-semi-simple sequent. We shall prove below that given such a dinat ρ whose associated proof-structure has a simple oriented cycle, the **Coh** dinat $\mathcal{J}_\omega \circ \mathcal{I}(\rho)$ would fail to be a morphism for some instantiation from **Coh**.

6.1 Main Theorem

The main theorem Theorem 6.2 below states that the proof-structure associated with a **GHCoh** -dinat is a MALL proof-net. Before beginning the proof, let us outline the approach we shall follow.

By the methods of functorial polymorphism [5], we may interpret formulas as multivariant functors, and proofs as dinatural transformations. The set of dinats interpreting the proofs of a sequent $\vdash \Delta$ is called the *proof space of* $\vdash \Delta$ and denoted by $\mathcal{PRF}(\vdash \Delta)$. We have the following inclusion

$$\mathcal{PRF}(\vdash \Delta) \quad \subseteq \quad \mathit{Dinat}\text{-}\mathcal{C}(\mathbf{1}, \Delta). \tag{5}$$

This holds for provable sequents $\vdash \Delta$ by the Soundness theorem, and for unprovable sequents $\vdash \Delta$, the proof space is empty; hence the result holds trivially.

If we strengthen (5) to equality, we obtain *full completeness* (for a given class of dinats), i.e.

$$\mathcal{PRF}(\vdash \Delta) \quad = \quad \mathit{Dinat}\text{-}\mathcal{C}(\mathbf{1}, \Delta). \tag{6}$$

In the main theorem we are interested in proving equalities of the form (6). The proof method of the main theorem (Theorem 6.2) works independently of whether the type of the dinat is provable or not. In outline, our method for proving (6) is the following:

(i) Suppose there is a **GHCoh** dinat ρ_0 outside the proof space.

(ii) Recall from Lemma 3.13 that there is a faithful functor $\mathcal{I}$: *Dinat*-**GHCoh** $\hookrightarrow$ *Dinat*-**HCoh** . We know from Corollary 4.55 that all **HCoh**-dinats ρ have an associated MALL proof-structure Θ_ρ (more generally, this is true for $\mathcal{C}$-dinats, for any $\mathcal{C}$ of Proposition 4.16). Thus via the embedding $\mathcal{I}$, the dinat ρ_0 has an associated MALL proof-structure Θ_{ρ_0}. Moreover we know from Lemma 5.16 that $(\Theta_{\rho_0})_S$ is connected for all normal switchings S. From (i), Θ_{ρ_0} is not a MALL net, hence by Proposition 5.15, Θ_{ρ_0} has a simple oriented canonical cycle with no critical jump.

(iii) Recall from Proposition 3.6 and Lemma 3.13 that there is a composition of faithful functors $\mathcal{J}_\omega \circ \mathcal{I}$: *Dinat*-**GHCoh** $\hookrightarrow$ *Dinat*-**Coh**. $\mathcal{J}_\omega \circ \mathcal{I}(\rho_0)$ is a **Coh**-dinat. We will construct a list of objects $\underline{\mathcal{A}} \in$ **Coh** such that $(\mathcal{J}_\omega \circ \mathcal{I}(\rho_0))_{\underline{\mathcal{A}}}$ is *not* a **Coh** morphism. This immediately leads to a contradiction.

Before beginning the main proof, we first illustrate this outline with an example.

Example 6.1 (Proof technique of the main theorem) Consider the example where the type Δ (of dinat ρ) is given by:

$$\vdash A \otimes ((C^\perp \parr C) \& (D^\perp \parr D)), B^\perp \oplus B^\perp, B \otimes A^\perp$$

We shall show Equation (6) for this choice of $\vdash \Delta$, which happens to be a provable sequent.

Now suppose for contradiction that there exists a dinat $\rho : \mathbf{1_G} \to \Delta$ in **GHCoh** which does not belong to the proof space of $\vdash \Delta$ (here we consider Δ as a multivariant functor). From Corollary 4.55 we can associate to ρ a MALL proof-structure Θ_ρ. Since ρ is not the

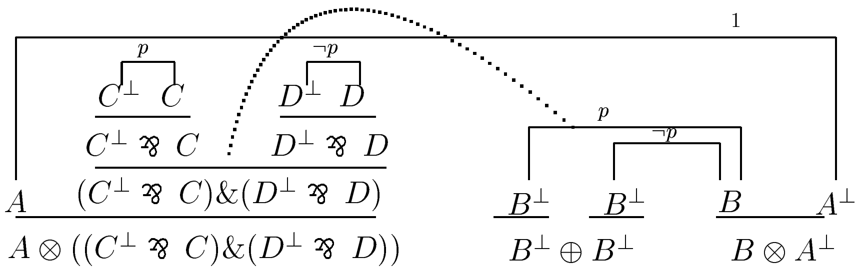


Figure 3: proof-structure Θ_ρ

denotation of a proof, Θ_ρ cannot be a proof-net, hence must have a cycle for some switching S, by Lemma 5.16. For example consider the case of the cycle in Figure 3, where p denotes the eigenweight for the unique &-link.

The dinat $\mathcal{J}_\omega \circ \mathcal{I}(\rho)$ determining the structure Θ_ρ is given by

$$(\mathcal{J}_\omega \circ \mathcal{I}(\rho))_{\mathcal{ABCD}} = \left\{ \begin{array}{c} (a,(1,(c,c)),(1,b),(b,a)) \\ (a,(2,(d,d)),(2,b),(b,a)) \end{array} \middle| \begin{array}{cc} a \in |\mathcal{A}| & c \in |\mathcal{C}| \\ b \in |\mathcal{B}| & d \in |\mathcal{D}| \end{array} \right\} \in (\Delta_{\mathcal{ABCD}})_p \tag{7}$$

In the above, $(\mathcal{J}_\omega \circ \mathcal{I}(\rho))_{\mathcal{ABCD}}$ and $\Delta_{\mathcal{ABCD}}$ denote the associated values at the objects $\mathcal{ABCD}$ of **Coh** as a subcategory of **GRel** (cf. Proposition 2.14). We shall show that $(\mathcal{J}_\omega \circ \mathcal{I}(\rho))_{\mathcal{ABCD}}$ is not a **Coh** morphism under the instantiation

$$\mathcal{A} = \mathcal{B} := (\{a_1,a_2\}, \{\emptyset, \{a_1\}, \{a_2\}\}, \{\emptyset, \{a_1\}, \{a_2\}, \{a_1,a_2\}\}) \in \mathbf{Coh}$$

where $\mathcal{C}$ and $\mathcal{D}$ are instantiated by arbitrary objects .
On the one hand, by taking $\tau := \left\{ \begin{array}{c} (a_1,(1,(c,c))), \\ (a_2,(2,(d,d))) \end{array} \middle| \begin{array}{c} c \in |\mathcal{C}| \\ d \in |\mathcal{D}| \end{array} \right\}$, we have

$$\tau \in Hom(\mathcal{A}, (\mathcal{C}^\perp ⅋ \mathcal{C})^\perp \oplus (\mathcal{D}^\perp ⅋ \mathcal{D})^\perp) = (\mathcal{A} \otimes ((\mathcal{C}^\perp ⅋ \mathcal{C}) \& (\mathcal{D}^\perp ⅋ \mathcal{D})))_{cp}. \tag{8}$$

On the other hand, by choosing $\{a_1\}$ (respectively $\{a_2\}$) belonging to the left (respectively the right) $\mathcal{B}_p$ in the equation below, we have

$$\{(1,a_1),(2,a_2)\} := \{\{a_1\}\}\frown\{\{a_2\}\} \in \mathcal{B}_p \frown \mathcal{B}_p = (\mathcal{B}^\perp \oplus \mathcal{B}^\perp)_{cp}.$$

Given that $\mathcal{B} = \mathcal{A}$, we have

$$id_{|\mathcal{A}|} \in Hom(\mathcal{B}, \mathcal{A}) = (\mathcal{B} \otimes \mathcal{A}^\perp)_{cp},$$

and thus, we can define an element δ as follows:

$$\delta := \left\{ \begin{array}{c} ((1,a_1),(a,a)) \\ ((2,a_2),(a,a)) \end{array} \middle| a \in |\mathcal{A}| \right\} = \{(1,a_1),(2,a_2)\} \times id_{|\mathcal{A}|} \in ((\mathcal{B}^\perp \oplus \mathcal{B}^\perp) ⅋ (\mathcal{B} \otimes \mathcal{A}^\perp))_{cp} \tag{9}$$

Now, from (8) and (9), we can construct a copoint $\eta = \tau \times \delta \in (\Delta_{\mathcal{ABCD}})_{cp}$ by

$$\eta := \left\{ \begin{array}{c} (a_1,(1,(c,c)),(1,a_1),(a,a)) \\ (a_1,(1,(c,c)),(2,a_2),(a,a)) \\ (a_2,(2,(d,d)),(1,a_1),(a,a)) \\ (a_2,(2,(d,d)),(2,a_2),(a,a)) \end{array} \middle| a \in |\mathcal{A}|, c \in |\mathcal{C}|, d \in |\mathcal{D}| \right\}$$

Now observe that $\#((\mathcal{J}_\omega \circ \mathcal{I}(\rho))_{\mathcal{ABCD}} \cap \eta) \geq 2$ which contradicts Proposition 2.14. □

Now we are ready to prove the Main Theorem.

Theorem 6.2 (Main Theorem) *Let σ be a dinat in* **GHCoh** *and $\Theta_\sigma \in PS(\sigma)$ be an associated proof-structure for σ as defined in Corollary 4.55. Then $(\Theta_\sigma)_S$ is acyclic for every switching and connected for every normal switching. Thus Θ_σ is a proof-net for* MALL.

Proof:

Suppose, for contradiction, that Θ is not a MALL proof-net. We may assume by Proposition 5.27 that the type of Θ is &-semi-simple. Then Lemma 5.16 assures the connectedness of $(\Theta_\sigma)_S$ for all normal switchings S. Hence by Proposition 5.15, Θ must have a simple oriented canonical cycle. Note that a **GHCoh**-dinat is a **HCoh**-dinat via the embedding $\mathcal{I}$: *Dinat*-**GHCoh** $\hookrightarrow$ *Dinat*-**HCoh** of Lemma 3.13. Thus $\mathcal{I}(\sigma)$ is an element of the set $\mathcal{S}$ of Lemma 5.17. Hence Lemma 5.17 implies that there exists a pair (ρ, Θ) of a dinat $\rho \in \mathcal{S}$ and a proof structure $\Theta \in PS(\rho)$ satisfying the following:

$$\text{Every simple oriented cycle in } \Theta \in PS(\rho) \text{ is global.} \tag{10}$$

Our goal is to show that this ρ fails to be a dinatural transformation for **HCoh**. But this is equivalent via the canonical embedding $\mathcal{J}_\omega$: *Dinat*-**HCoh** $\hookrightarrow$ *Dinat*-**Coh** (cf. Proposition 3.6) to showing that $\mathcal{J}_\omega(\rho)$ fails to be a dinatural transformation of **Coh**. For this we shall prove that for some instantiation $\underline{A}$ in **Coh**, $(\mathcal{J}_\omega(\rho))_{\underline{A}}$ is not a **Coh** morphism.

$(\Theta_\rho)_S$ has a simple oriented canonical cycle of the form in Figure 2 together with Definition 5.6. We may assume without loss of generality that $\mathtt{C}$ appears under the valuation φ_S such that $\varphi_S(p_i) = 1$ for all $i = 1, \ldots, n$. Hence under the assumption, simplicity of $\mathtt{C}$ means that for all $i = 1, \ldots, n$, $w(\mathtt{ax}_{i+1}) = p_i.v_i \pmod n$ where v_i does not depend on any p_j $(1 \leq j \leq n)$. Indeed, we have this $v_i = 1$ by virtue of (10) and Lemma 5.10. Then the local shape of $\mathtt{C}$ around the $(i-1)$-th jump is given in the following Figure 4:

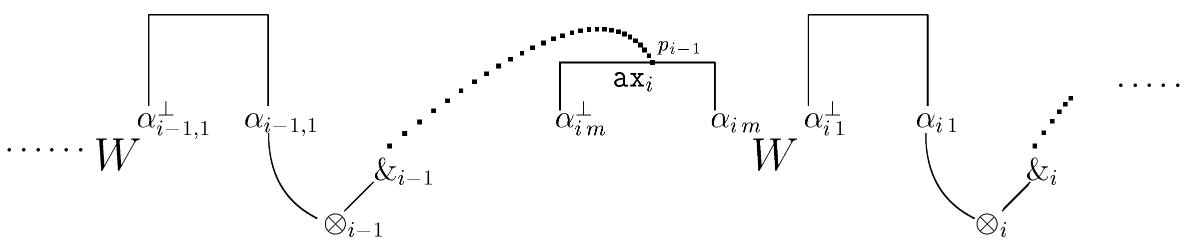


Figure 4: The shape of the graph $(\Theta_\rho)_S$

Note that in Figure 4, α_{lk} denotes a literal. $\otimes$-formula in $(\Theta_\rho)_S$ is a hereditary conclusion (using only $\otimes$ links) of $\alpha_{i-1,1}$ and the immediate conclusion of the $\&_{i-1}$ formula. Again by semi-simplicity, there must be a path which we denote by W between $\alpha_{i\,m}$ and $\alpha^\perp_{i\,1}$ which uses only $\otimes$-links and axiom-links.

Our first task is to determine the form of the morphism ρ, given the above (simple oriented) cycle. We claim the proof-structure Θ_ρ must be of the following form (see Figure 5):

Note first that in Θ_ρ all links between $\alpha_{i-1,1}$ and $\otimes_{i-1}$ are $\otimes$-links by the assumption of semi-simplicity of the type of Θ. Hence $w(\alpha_{i-1,1}) = w(L_{i-1})$ holds from the unique link property

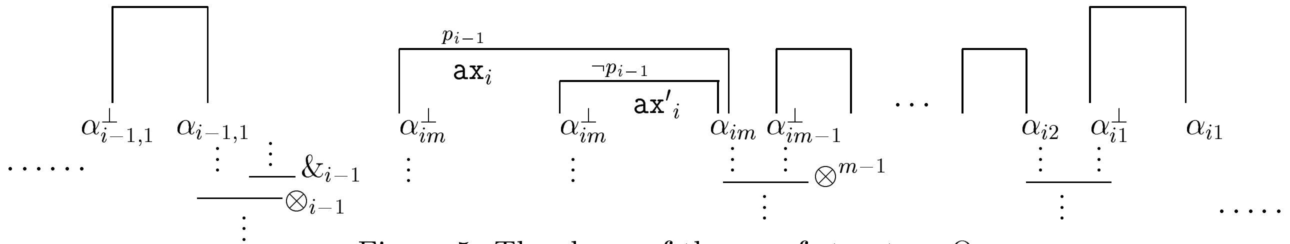


Figure 5: The shape of the proof-structure Θ_ρ

of Corollary 4.55, where L_{i-1} is the $(\&_{i-1})$-link of Figure 5. On the other hand from Lemma 5.10 and (10), we have $w(L_{i-1}) = 1$. (Of course the same situation holds around the i-th jump; i.e., $w(\alpha_{i1}) = w(L_i) = 1$ $(1 \leq j \leq n)$.)

In Figure 5, the $(m-1)$ $\otimes$-links $\otimes^1, \ldots, \otimes^{m-1}$ are the outermost connectives of the path W. Thus the link $\otimes^k$ is hereditarily below both α_{ik} and $\alpha^\perp_{ik+1}$, and all links between α_{ik} (respectively $\alpha^\perp_{ik+1}$) and $\otimes^k$ are $\otimes$-links by the assumption of semi-simplicity. Thus from the unique link property, we have $w(\alpha_{im}) = \cdots = w(\alpha_{i2}) = w(\alpha_{i1}) = w(L_i) = 1$ $(1 \leq j \leq n)$ from the above.

If we change the switching from S to S' so that the valuation $\varphi_S(p_{i-1}) = 1$ changes to $\varphi_{S'}(p_{i-1}) = 0$, the $(\mathtt{ax}_i)$-link in Figure 4 disappears. But the formula occurrence α_{im} remains in the graph ($w(\alpha_{im})$ does not depend on p_{i-1} because $w(\alpha_{im}) = 1$) and, being a literal, is the conclusion of some axiom link. Hence in the proof-structure Θ_ρ, there must exist two axiom links whose conclusion is the occurrence α_{im}: one is the $\mathtt{ax}_i$ appearing in the graph $(\Theta_\rho)_S$, whose weight is p_{i-1}, and the other, say $\mathtt{ax}'_i$, which does not appear in the graph $(\Theta_\rho)_S$ (instead it appears in $(\Theta_\rho)_{S'}$). The weight of this axiom link $\mathtt{ax}'_i$ has an occurrence $\neg p_{i-1}$. In fact we shall show that $w(\mathtt{ax}'_i)$ is exactly equal to $\neg p_{i-1}$. First, in $(\Theta_\rho)_{S'}$ we can draw a jump from $\&_{i-1}$ to $\mathtt{ax}'_i$. Second, for all $j \in \{1, \ldots, n\} \setminus \{i-1\}$, all the jumps in $\mathtt{C}$ from $\&_j$ to $\mathtt{ax}_{j+1}$ are retained in $(\Theta_\rho)_{S'}$ since both $\&_j$ and $\mathtt{ax}_{j+1}$ occur under S' by noting that $w(L_j) = 1$ and $w(\mathtt{ax}_{j+1}) = p_j$. This yields a simple oriented cycle in $(\Theta_\rho)_{S'}$. Thus from Lemma 5.10 and the above (10), we have $w(\mathtt{ax}'_i) = \neg p_{i-1}$.

From $w(\mathtt{ax}_i) = p_{i-1}$ and $w(\mathtt{ax}'_i) = \neg p_{i-1}$, we have that the two $\alpha^\perp_{im}$'s which are conclusions of $\mathtt{ax}_i$ and $\mathtt{ax}'_i$ are different occurrences, by virtue of the no duplicate axiom-link property in Θ_ρ.

Since **Coh** validates Mix , we apply $\mathsf{Mix} : X \otimes Y \to X \parr Y$ to all $\otimes$ occurrences in the type of $\mathcal{J}_\omega(\rho)$ *except* those lying on the cycle $\mathtt{C}$ of Figure 4. Note that this process does not affect the cycle. As for the cycle $\mathtt{C}$ itself, by commutativity and associativity of tensor, we may assume $\otimes_{i-1}$ is immediately below $\alpha_{i-1,1}$. Thus we obtain a **Coh** dinat $\widetilde{\mathcal{J}_\omega(\rho)} \colon \mathbf{1} \to \tilde{\Delta}$ whose type $\vdash \tilde{\Delta}$ is the following sequent:

$$\begin{gathered}\vdash \cdots \alpha_{i-1,1} \otimes (B_1 \& B_2),\ N[\, \alpha^\perp_{im}, \alpha^\perp_{im} \,], \\ \alpha_{im} \otimes \alpha^\perp_{im-1},\ \ldots, \alpha_{i3} \otimes \alpha^\perp_{i2},\ \alpha_{i2} \otimes \alpha^\perp_{i1},\ \Xi \cdots\end{gathered} \tag{11}$$

where

- $N[\, *_1, *_2 \,]$ is either $*_1 \parr *_2$, $N_1[\, *_1 \,] \oplus N_2[\, *_2 \,]$ or $(N_1[\, *_1 \,] \oplus N'_1) \parr (N'_2 \oplus N_2[\, *_2 \,])$ with all connectives in N_i being $\parr$ $(i = 1, 2)$. Note that in $N[\, \alpha^\perp_{im}, \alpha^\perp_{im} \,]$ we instantiate twice with the same literal $\alpha_{im}^{\perp}$, because of its two distinct occurrences in the above Figure 5.

- Ξ is $E_{11} \oplus E_{12}, \ldots, E_{m1} \oplus E_{m2}, \ell_1, \ldots, \ell_r$
  with all connectives in E_{lj} being ⅋ and ℓ_r being literals.

In what follows we instantiate all atoms occurring in $\tilde{\Delta}$ by a single object $\mathcal{A} \in \mathbf{Coh}$: i.e., we consider a morphism

$$(\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}} \colon \mathbf{1} \to \tilde{\Delta}_{\underline{\mathcal{A}}}, \quad \text{equivalently} \quad (\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}} \in (\tilde{\Delta}_{\underline{\mathcal{A}}})_p \;. \tag{12}$$

Now every element of $(\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}}$ is of the following form:

$$\cdots((x_{i-1,1}, \underline{y_1}, (k, \quad)), (\quad, x_{im}), (x_{im}, \underline{y_m}, x_{im-1}), \ldots, (x_{i3}, \underline{y_3}, x_{i2}), (x_{i2}, \underline{y_2}, x_{i1}), \_, \_)\cdots \tag{13}$$

where $k \in \{1, 2\}$ denotes the first/second component of $B_1 \& B_2$ and $\forall i \in \{1, \ldots, n\}$ x_i and $\underline{y_i}$ are arbitrary elements of $\mathcal{A}$ and $\underline{\mathcal{A}}$ respectively.

Our next task is to construct an $\eta \in (\tilde{\Delta}_{\underline{\mathcal{A}}})_{cp}$ for the morphism $(\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}} \in (\tilde{\Delta}_{\underline{\mathcal{A}}})_p$ so that we can derive a contradiction. For this purpose we prove the following instantiation lemma, which is crucial in our proof of acyclicity:

Lemma 6.3 (Instantiation Lemma) *We instantiate (12) above as follows.*

$$\mathcal{A} := \left(\left\{ \begin{array}{cc} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{array} \right\},\; P_{<2}(|\mathcal{A}|) \cup \left\{ \begin{array}{c} \{a_{11}, a_{12}\} \\ \{a_{21}, a_{22}\} \\ \{a_{11}, a_{22}\} \end{array} \right\},\; P_{<2}(|\mathcal{A}|) \cup \left\{ \begin{array}{c} \{a_{11}, a_{21}\} \\ \{a_{12}, a_{22}\} \\ \{a_{12}, a_{21}\} \end{array} \right\} \right) \in \mathbf{Coh}$$

Note that $\mathcal{A} \cong \mathcal{A}^{\perp}$ via the cyclic permutation $g := (a_{11}, a_{21}, a_{22}, a_{12})$ on $|\mathcal{A}|$. Then the following properties hold, where $\mathcal{A}_{lk}$ and $\mathcal{B}_k$ denote the objects resulting respectively from α_{lk} and B_k of (11) by the instantiation (thus each $\mathcal{A}_{lk}$ is $\mathcal{A}$ or $\mathcal{A}^{\perp}$):

(i) For fixed $\beta_1 \in (\mathcal{B}_1)_{cp}$ and $\beta_2 \in (\mathcal{B}_2)_{cp}$, define $\tau_{\beta_1\beta_2}$ as follows:

$$\tau_{\beta_1\beta_2} := \left\{ \begin{array}{c} (a_{11}, (1, b_1)), \\ (a_{21}, (2, b_2)) \end{array} \;\middle|\; \begin{array}{c} b_1 \in \beta_1 \\ b_2 \in \beta_2 \end{array} \right\}.$$

Then we have

$$\tau_{\beta_1\beta_2} \in Hom(\mathcal{A}, \mathcal{B}_1^{\perp} \oplus \mathcal{B}_2^{\perp}) := (\mathcal{A} \otimes (\mathcal{B}_1 \& \mathcal{B}_2))_{cp}$$

(i') For fixed $\beta_1 \in (\mathcal{B}_1)_{cp}$ and $\beta_2 \in (\mathcal{B}_2)_{cp}$, define $\tau_{\beta_1\beta_2}$ as follows:

$$\tau_{\beta_1\beta_2} := \left\{ \begin{array}{c} (a_{11}, (1, b_1)), \\ (a_{12}, (2, b_2)) \end{array} \;\middle|\; \begin{array}{c} b_1 \in \beta_1 \\ b_2 \in \beta_2 \end{array} \right\}.$$

Then we have

$$\tau_{\beta_1\beta_2} \in Hom(\mathcal{A}^{\perp}, \mathcal{B}_1^{\perp} \oplus \mathcal{B}_2^{\perp}) := (\mathcal{A}^{\perp} \otimes (\mathcal{B}_1 \& \mathcal{B}_2))_{cp}$$

(ii) Let us define ι by

$$\iota := \{(g_{l-1}(a), a) \mid a \in |\mathcal{A}|\} \quad \textit{where} \quad g_{l-1} = \begin{cases} id_{|\mathcal{A}|} & \textit{if } \mathcal{A}_{il} = \mathcal{A}_{il-1} \\ g & \textit{if } \mathcal{A}_{il} = \mathcal{A}^{\perp}_{il-1}. \end{cases}$$

Then we have

$$\iota \in Hom(\mathcal{A}_{il}, \mathcal{A}_{il-1}) = (\mathcal{A}_{il} \otimes \mathcal{A}^{\perp}_{il-1})_{cp}$$

Proof of Lemma 6.3. We shall prove only (i) ((i)'and (ii) are similar). In the following, $\tau_{\beta_1\beta_2}$ is abbreviated simply to τ. At this point, the reader should recall the definitions of the image and coimage conditions on morphisms in **GRel** (cf. Definition 2.13). We verify:

(*image condition on* τ) For an arbitrary $\emptyset \neq s \in \mathcal{A}_p$, we have $\{a_{11}, a_{21}\} \not\subseteq s$. Hence either $[s]\tau = \beta_1, [s]\tau = \beta_2$ or $[s]\tau = \emptyset$, which implies $[s]\tau \in (\mathcal{B}_1)_{cp} + (\mathcal{B}_2)_{cp} = (\mathcal{B}_1^{\perp} \oplus \mathcal{B}_2^{\perp})_p$.

(*co-image condition on* τ) Take an arbitrary $r \in (\mathcal{B}_1^{\perp} \oplus \mathcal{B}_2^{\perp})_{cp} = (\mathcal{B}_1^{\perp})_{cp} \frown (\mathcal{B}_2^{\perp})_{cp} = (\mathcal{B}_1)_p \frown (\mathcal{B}_2)_p$. Then we have the following:

$$\tau[r] = \begin{cases} \emptyset & \text{if } r_1 \cap \beta_1 = \emptyset \text{ and } r_1 \cap \beta_1 = \emptyset \\ \{a_{11}\} & \text{if } r_1 \cap \beta_1 \neq \emptyset \text{ and } r_1 \cap \beta_1 = \emptyset \\ \{a_{21}\} & \text{if } r_1 \cap \beta_1 = \emptyset \text{ and } r_1 \cap \beta_1 \neq \emptyset \\ \{a_{11}, a_{21}\} & \text{if } r_1 \cap \beta_1 \neq \emptyset \text{ and } r_1 \cap \beta_1 \neq \emptyset \end{cases}$$

In all cases we have $\tau[r] \in \mathcal{A}_{cp}$. □

Now we are ready to construct the set η:

First by (11) and (13) we can take two distinct elements c_1 and c_2 from $(\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}}$;

$$c_1 = \cdots ((a_{11}, \underline{y_1}, (1, b^1)), (m^1, a_{11}^{m-1}), (a_{11}^{m-1}, \underline{y_m}, a_{11}^{m-2}), \ldots, (a_{11}^2, \underline{y_3}, a_{11}^1), (a_{11}^1, \underline{y_2}, a_{11}), \underline{u^1}, \underline{v}) \cdots$$

$$c_2 = \cdots ((\tilde{a}_{i-1}, \underline{y_1}, (2, b^2)), (m^2, \tilde{a}_i^{m-1}), (\tilde{a}_i^{m-1}, \underline{y_m}, \tilde{a}_i^{m-2}), \ldots, (\tilde{a}_i^2, \underline{y_3}, \tilde{a}_i^1), (\tilde{a}_i^1, \underline{y_2}, \tilde{a}_i), \underline{u^2}, \underline{v}) \cdots$$

where

- $\tilde{a}_j = \begin{cases} a_{21} & \text{if } \alpha_{j\,1} \text{ is an atom} \\ a_{12} & \text{if } \alpha_{j\,1} \text{ is a negation of an atom} \end{cases}$ $\qquad j \in \{1, \ldots, i-1, i, \ldots, n\}$
- $a^r = g_r \circ \cdots \circ g_1(a)$ for $a \in |\mathcal{A}|$ (cf. Lemma 6.3 (ii) for definitions of g_r).
- b^1 is chosen such that there exists $\beta_1 \in (\mathcal{B}_1)_{cp}$ with $b^1 \in \beta_1$ and similarly for b^2.
- The two pairs $(m^1, a_{11}), (m^2, a^i)$ are chosen such that there exists $\delta \in (\mathcal{N}[\, \mathcal{A}^{\perp}_{im}, \mathcal{A}^{\perp}_{im}\,])_{cp}$ with $\{(m^1, a_{11}), (m^2, a^i)\} \in \delta$. We let (m^j, x) denote x when $\mathcal{N}[\, *_1, *_2\,]$ is of the form $*_1 ⅋ *_2$.
- $\underline{u^1}$ and $\underline{u^2}$ are vectors respectively of u_j^1 and u_j^2 such that $\exists\, \epsilon_{j1} \in (\mathcal{E}_{j1})_{cp}\ u_j^1 \in \epsilon_{j1}$ and $\exists\, \epsilon_{j2} \in (\mathcal{E}_{j2})_{cp}\ u_j^2 \in \epsilon_{j2}$, where $\mathcal{E}_{ji}$ is the instantiation of E_{ji}.
- We choose $\underline{v}$ such that there exists $\xi \in (\mathcal{L}_1 ⅋ \cdots ⅋ \mathcal{L}_r)_{cp} = (\mathcal{L}_1)_{cp} \times \cdots \times (\mathcal{L}_r)_{cp}$ such that $\underline{v} \in \xi$, where $\mathcal{L}_k$ is the instantiation of ℓ_k (i.e., $\mathcal{L}_k$ will be $\mathcal{A}$ or $\mathcal{A}^{\perp}$).

Second from (11) and the fact that $(\mathcal{X} \mathbin{⅋} \mathcal{Y})_{cp} = \mathcal{X}_{cp} \times \mathcal{Y}_{cp}$, we have

$$\begin{array}{rl}(\tilde{\Delta}_{\underline{\mathcal{A}}})_{cp} = & \cdots (\mathcal{A}_{i-1,1} \otimes (\mathcal{B}_1 \& \mathcal{B}_2))_{cp} \times (\mathcal{N}[\,\mathcal{A}_{im}^{\perp},\, \mathcal{A}_{im}^{\perp}\,])_{cp} \\ & \times (\mathcal{A}_{im} \otimes \mathcal{A}_{i,m-1}^{\perp})_{cp} \times \cdots \times (\mathcal{A}_{i2} \otimes \mathcal{A}_{i1}^{\perp})_{cp} \times (\Xi_{\underline{\mathcal{A}}})_{cp} \cdots\end{array}$$

Thus by taking δ, ϵ_{ji} and ξ as above and $\tau_{\beta_1\beta_2}$ and ι as in Lemma 6.3, we define $\eta \in (\tilde{\Delta}_{\underline{\mathcal{A}}})_{cp}$ by

$$\eta = \cdots \tau_{\beta_1\beta_2} \times \delta \times \iota \times \cdots \iota \times (\epsilon_{11} + \epsilon_{12}) \times \cdots \times (\epsilon_{m1} + \epsilon_{m2}) \times \xi \cdots$$

Here $\tau_{\beta_1\beta_2}$ is taken from (i) or (i') of Lemma 6.3 according to whether $\alpha_{i-1,1}$ is an atom or the negation of an atom, respectively.

From the construction, we have $c_1, c_2 \in \eta$. Thus

$$\#((\widetilde{\mathcal{J}_\omega(\rho)})_{\underline{\mathcal{A}}} \cap \eta) \geq 2$$

This contradicts Proposition 2.14, since all atoms are instantiated at the coherence space $\mathcal{A}$. □

We thus immediately conclude the main result of our paper:

Theorem 6.4 *Dinat*-**GHCoh** *is fully complete for* MALL.

7 Remarks on the Mix rule

Previously in the paper, we have made substantial use of the theory ⅋ALL +Mix, in particular the ⅋ALL +Mix full completeness of *Dinat*-**HCoh** (Corollary 4.2). In this section, we consider the full theory MALL+Mix . One might expect that *Dinat*-**HCoh** is fully complete for this theory. Despite the fact that the category *Dinat*-**HCoh** has the strong properties of *softness* and MLL+Mix full completeness, this is not the case. Indeed, we show that *Dinat*-**HCoh** is *not* MALL+Mix fully complete. This suggests that MALL+Mix is a more complex theory than MALL, in sharp contrast to the purely multiplicative case.

A counterexample is given by the following:

Define a family $\rho = \{\rho_{ABCDEF} \mid A, B, C, D, E, F \in \mathbf{HCoh}\}$ by

$$\rho_{ABCDEF} = \left\{ \begin{array}{l} ((1,(a,b)), a, (1,b)) \\ ((2,(a,c)), a, (2,c)) \end{array} \right\}_{a\in|A|, b\in|B|, c\in|C|} \times \left\{ \begin{array}{l} ((1,e), d, (1,(e,d))) \\ ((2,f), d, (2,(f,d))) \end{array} \right\}_{d\in|D|, e\in|E|, f\in|F|}$$

Then we have

Proposition 7.1 *ρ becomes a dinat of* **HCoh**, *whose type $\vdash \Delta$ is given by the following* MALL+Mix *sequent:*

$$\vdash\ (A^\perp ⅋ B^\perp) \oplus (A^\perp ⅋ C^\perp),\ A \otimes D,\ (B \& C) \otimes (E \& F),\ (E^\perp ⅋ D^\perp) \oplus (F^\perp ⅋ D^\perp)$$

Moreover ρ is not the denotation of any MALL+Mix *proof.*

Proof. First we shall check that every member of the family is a morphism of **HCoh**. For this, given an arbitrary $u \subseteq^*_{fin} \rho_{ABCDEF}$, we shall prove that

$$u \in \Gamma(\Delta) \quad \text{where } \Gamma(X_1, \dots, X_n) = \Gamma(X_1 \parr \cdots \parr X_n) \tag{14}$$

Since $\pi_3(u) \in \Gamma((B\&C) \otimes (E\&F))$ directly implies (14), we assume

$$\pi_3(u) \notin \Gamma((B\&C) \otimes (E\&F)) \quad \text{or equivalently} \quad \pi_1(\pi_3(u)) \notin \Gamma(B\&C) \text{ or } \pi_2(\pi_3(u)) \notin \Gamma(E\&F)$$

By the symmetry of B and C with respect to E and F, without loss of generality, we may assume that

$$w := \pi_1(\pi_3(u)) \notin \Gamma(B\&C) \quad \text{or equivalently}$$
$$(w_2 = \emptyset \text{ and } w_1 \notin \Gamma(B)) \quad \text{or} \quad (w_1 = \emptyset \text{ and } w_2 \notin \Gamma(C))$$

Again, by the symmetry of B with respect to C, we may assume that

$$w_2 = \emptyset \wedge w_1 \notin \Gamma(B) \quad \text{or equivalently} \quad w_2 = \emptyset \wedge w_1 \in \Gamma(B^\perp) \tag{15}$$

On the other hand, the definition of ρ_{ABCDEF} implies the following:

$$\text{If } (\pi_1(\pi_3(u)))_2 = w_2 = \emptyset \quad \text{then } (\pi_1(u))_2 = \emptyset.$$
$$\text{Also we have } w_1 = \pi_2((\pi_1(u))_1).$$

The above facts, together with (15), imply

$$(\pi_1(u))_2 = \emptyset \quad \text{and} \quad (\pi_1(u))_1 \in \Gamma(A^\perp \parr B^\perp)$$

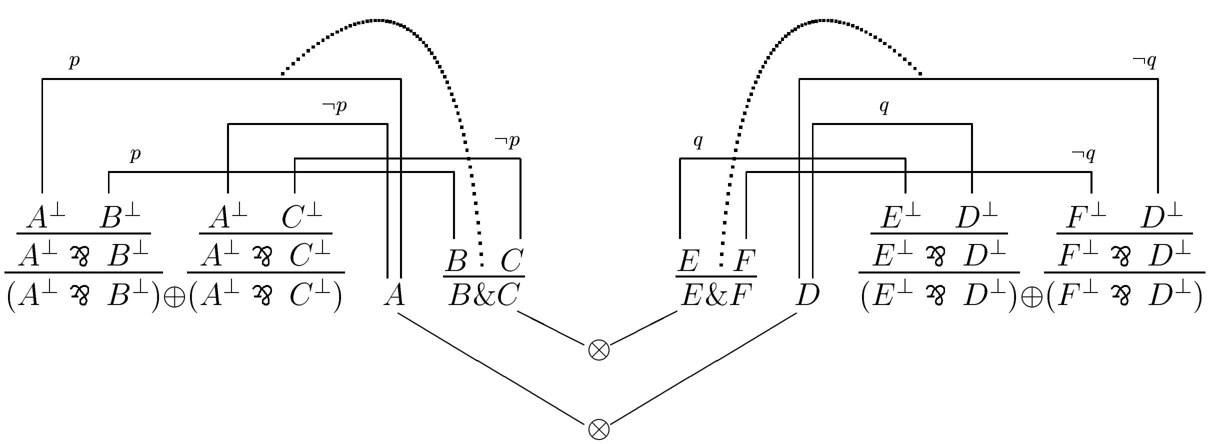
But these imply $\pi_1(u) \in \Gamma((A^\perp \parr B^\perp) \oplus (A^\perp \parr C^\perp))$, thus we have (14).

Second we check that the family is a dinatural transformation. ρ happens to be a denotation of a MALL+Mix proof of the following type, which is obtained from Δ by erasing the two outermost tensors:

$$\vdash \ (A^\perp \parr B^\perp) \oplus (A^\perp \parr C^\perp),\ A, D,\ B\&C, E\&F,\ (E^\perp \parr D^\perp) \oplus (F^\perp \parr D^\perp)$$

Hence, by soundness of the dinatural interpretation, ρ is a dinat of the latter type, thus it is a dinat of the original type Δ as well.

Finally the MALL proof-structure Θ_ρ associated to the dinat ρ (cf. Corollary 4.55) is given by the following, which has a cycle C (with two jumps). Hence Θ_ρ is not a MALL+Mix proof-net.



where p and q are respective eigenweights for the left and right &-links. $\square$

In contrast to the MALL case (without MIX), the cycle above is unoriented.

8 Conclusion

This paper establishes a non-game-theoretic dinatural full completeness theorem for MALL in the double gluing category **GHCoh** . A key ingredient is Joyal's notion of softness, which equates dinaturality with Girard's MALL proof-structures. Along the way, this involves a careful analysis of several interesting subtheories, and certain restrictions on allowable proof structures. In particular, an analysis of the possible shapes of cycles in non-nets is developed.

Typically in proving a full completeness theorem, one would also wish to verify faithfulness of the interpretation. However we have not proved that and leave it as an open problem. This is related to the fact that there is no known precise correspondence between MALL proof-nets and the free $*$-autonomous category with products, unlike in the purely multiplicative case [8]. Such a correspondence was exploited by the authors in their various MLL full completeness theorems [9, 10, 23, 24].

Hughes and van Glabbeek [27] consider a larger class of MALL proof structures by eliminating the restriction of Girard's dependency condition. For this class, Girard's original correctness condition is insufficient. They thus introduce a stronger correctness criterion for distinguishing MALL proof nets.

The Hughes-van Glabbeek system of proof structures associates a unique proof-structure to each dinat, owing to the elimination of the dependency condition. Hence a promising direction for future work would be to investigate the possibility of a faithful full completeness theorem using this larger class of structures.

However this extension of our results to this larger class of structures might be difficult given that their criterion is not a canonical extension of Girard's. When their criterion is restricted to Girard's class of proof structures (with dependency condition), one obtains a *different* correctness criterion from Girard's.

Another problem we leave open is the question of finding other soft categories, besides **HCoh** and categories of games, which are models of MALL

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