Exponential Functions in a Cartesian Differential Category

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(Thanks for the invite Rick!)

Differential Categories: The Four Tomes

Differential Categories

Blute, Cockett, Seely - 2006

Cartesian Differential Categories

Blute, Cockett, Seely - 2009

Differential Restriction Categories

Cockett, Cruttwell, Gallagher - 2011

Tangent Categories

Rosicky - 1984

Cockett, Cruttwell - 2014

TODAY'S STORY: Generalizing the exponential function e^x to Cartesian Differential Categories.

Lemay, J-S. P. **Exponential Functions in Cartesian Differential Categories**. Applied Categorical Structures (2020). https://doi.org/10.1007/s10485-020-09610-0

Cartesian Differential Categories

A **Cartesian differential category** is:

- A category $\mathbb{X}$ with finite products
- Where each homset $\mathbb{X}(A, B)$ is a commutative monoid with:

$$+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B) \qquad\qquad 0 \in \mathbb{X}(A, B)$$

  such that composition preserves the addition in the following sense:

$$(f + g) \circ x = f \circ x + g \circ x \qquad\qquad 0 \circ x = 0$$

- and $\mathbb{X}$ comes equipped with a **differential combinator** D:

$$\frac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}$$

  such that D satisfies axioms which generalize the directional derivative from multivariable differential calculus such as the chain rule, linearity in its second argument, symmetry of the partial derivatives, etc. (there are 7 axioms).

R. Blute, R. Cockett, R.A.G. Seely, **Cartesian Differential Categories** , Theory and Applications of Categories 23, pp. 622-672, 2009

Main Example of a Cartesian Differential Category

Let SMOOTH be the category of smooth real functions, that is, the category whose objects are the Euclidean vector spaces $\mathbb{R}^n$ and whose maps are smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which is actually an m-tuple of smooth functions $F = \langle f_1, \ldots, f_n \rangle$, where $f_i : \mathbb{R}^n \to \mathbb{R}$.

Example

SMOOTH is a CDC where the differential combinator is given by the directional derivative of smooth functions. Explicitly, for a smooth map $f : \mathbb{R}^n \to \mathbb{R}$ recall that the gradient of f is:

$$\nabla(f) : \mathbb{R}^n \to \mathbb{R}^n \qquad \nabla(f)(\vec{v}) := \langle \frac{\partial f}{\partial x_1}(\vec{v}), \ldots, \frac{\partial f}{\partial x_n}(\vec{v}) \rangle$$

Then the differential combinator is defined as the directional derivative:

$$\mathsf{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \mathsf{D}[f](\vec{v}, \vec{w}) := \nabla(f)(\vec{v}) \cdot \vec{w} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{v}) w_i$$

For a smooth map $F : \mathbb{R}^m \to \mathbb{R}^n$, $F = \langle f_1, \ldots, f_n \rangle$, the differential combinator is defined as:

$$\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \qquad \mathsf{D}[F] := \langle \mathsf{D}[f_1], \ldots, \mathsf{D}[f_n] \rangle$$

which can also be defined using the Jacobian of F:

$$\mathsf{D}[F](\vec{v}, \vec{w}) = \mathsf{J}(F)(\vec{v}) \vec{w}^T$$

Other Examples of Cartesian Differential Categories

Example

- Any category with finite biproduct $\oplus$ is a CDC, where for a map $f : A \to B$:

$$\mathsf{D}[f] := \; A \oplus A \xrightarrow{\quad \pi_1 \quad} A \xrightarrow{\quad f \quad} B$$

  For example, for any commutative semiring R, the category of R-modules MOD_R is a CDC where $\mathsf{D}[f](x, y) = f(y)$.

- For any commutative semiring R, let POLY_R be the category whose objects are the natural numbers $n \in \mathbb{N}$ and where a map $P : n \to m$ is an m-tuple of polynomials $P := \langle p_1, \ldots, p_m \rangle$, where $p_i \in R[x_1, \ldots, x_n]$. POLY_R is a CDC where $\mathsf{D}[P] : n \times n \to m$ is defined as:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_n}{\partial x_i} y_i \right\rangle$$

  where $\sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} y_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-CDC of SMOOTH.

Other Examples of Cartesian Differential Categories

Example

- The coKleisli category of a differential category is a CDC (more on this example later)

  R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**, Mathematical Structures in Computer Science Vol. 1616, pp 1049-1083, 2006.

- Every model of the differential λ-calculus induces a CDC.

  Manzonetto, G., 2012. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. Mathematical Structures in Computer Science, 22(3), pp.451-520.

- Bauer et. al (BJORT) constructed an Abelian functor calculus model of a CDC.

  Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A., 2018. **Directional derivatives and higher order chain rules for abelian functor calculus.** Topology and its Applications, 235, pp.375-427.

- There exists both free and cofree constructions of CDCs.

  Cockett, J.R.B. and Seely, R.A.G., 2011. **The Faa di bruno construction**. Theory and applications of categories, 25(15), pp.394-425.

Exponential Function

The goal is now to generalize the exponential to Cartesian differential categories.

The exponential function $e^x : \mathbb{R} \to \mathbb{R}$ can be defined in numerous equivalent ways:

- The (partial) inverse of the natural logarithm function $ln(x)$
- As the limit:

$$e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n$$

- As the convergent power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- As the solution to $f'(x) = f(x)$ with initial condition $f(0) = 1$

PROBLEM: In arbitrary Cartesian differential categories, one does not necessarily have partial functions, a notion of convergence, infinite sums, or even (unique) solutions to initial value problems. So we must look for a more algebraic characterization of the exponential function.

Exponential Function

The exponential function $e^x : \mathbb{R} \to \mathbb{R}$ satisfies the following important properties:

$$e^{x+y} = e^x e^y \qquad\qquad e^0 = 1$$

These seem like reasonable equations one can write down in a CDC, so what's the problem?

PROBLEM: While this seems promising, arbitrary objects in a Cartesian differential category do not necessarily come equipped with a multiplication.

SOLUTION: Rather than requiring this extra ring structure on objects, it turns out that the differential combinator D will allow to bypass the need for a multiplication!

Directional Derivative of the Exponential Function

The directional derivative of the exponential function $e^x : \mathbb{R} \to \mathbb{R}$ is:

$$\mathsf{D}[e^x] : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad \mathsf{D}[e^x](a, b) = \frac{\partial e^x}{\partial x}(a) b = e^a b$$

so the multiplication of $\mathbb{R}$ appears!

We can express $e^{x+y} = e^x e^y$ and $e^0 = 1$ in terms of the differential combinator as follows:

$$\mathsf{D}[e^x](0, b) = e^0 b = b \qquad \mathsf{D}[e^x](a, e^b) = e^a e^b = e^{a+b}$$

and these equations we can easily generalize to a CDC.

Differential Exponential Maps

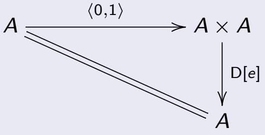
In a Cartesian differential category, for an object A define the map $\oplus : A \times A \to A$ as:

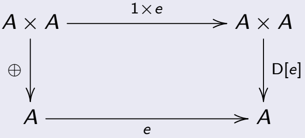
$$\oplus = \pi_0 + \pi_1$$

In SMOOTH, this gives the point-wise addition of vectors.

Definition

A **differential exponential map** in a Cartesian differential category is a map $e : A \to A$, such that the following diagrams commute:





Using element notation, the above diagrams are:

$$\mathrm{D}[e](0, x) = x \qquad \mathrm{D}[e](x, e(y)) = e(x + y)$$

Differential Exponential Maps – Examples in SMOOTH

Example

(i) The exponential function $e^x : \mathbb{R} \to \mathbb{R}$ is a differential exponential map.

(ii) The point-wise exponential function, $e^x \times e^y : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (e^x, e^y)$, is a differential exponential map.

(iii) Define the smooth function $\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (e^x \cos(y), e^x \sin(y))$. Then ϵ is a differential exponential map. But what is ϵ?

It's the complex exponential function: Writing as (x, y) as $x + iy$ with $i^2 = -1$:

$$e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

Differential Exponential Maps – Examples in SMOOTH

Example

- (iv) Define the smooth function $\epsilon' : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (e^x \cosh(y), e^x \sinh(y))$. Then ϵ is a differential exponential map. But what is ϵ'?

  It's the split-complex exponential function: Writing as (x, y) as $x + jy$ with $j^2 = 1$:

  $$e^{x+jy} = e^x \cosh(y) + je^x \sinh(y)$$

- (v) Define the smooth function $\mathsf{T}(e^x) : \mathbb{R}^2 \to \mathbb{R}^2$ as $(x, y) \mapsto (e^x, e^x y)$. Then $\mathsf{T}(e^x)$ is a differential exponential map. But what is this map?

  It's the dual numbers exponential function: Writing as (x, y) as $x + \varepsilon y$ with $\varepsilon^2 = 0$:

  $$e^{x+y\varepsilon} = e^x + e^x y\varepsilon$$

Example

There are no non-trivial differential exponential maps in POLY_R or a category with finite biproducts (i.e. the only differential exponential map is the identity of the terminal object, $1 = 0$).

Lemma

In a Cartesian differential category $\mathbb{X}$:

1. *For the terminal object $\top$, the identity $1_\top : \top \to \top$ is a differential exponential map;*
2. *If $e : A \to A$ and $e' : B \to B$ are differential exponential maps, then $e \times e' : A \times B \to A \times B$ is a differential exponential map;*
3. *Let $\mathsf{T} : \mathbb{X} \to \mathbb{X}$ be the tangent endofunctor defined as:*

$$\mathsf{T}(A) = A \times A \qquad \mathsf{T}(f) = \langle f \circ \pi_0, \mathsf{D}[f] \rangle$$

   Then if $e : A \to A$ is a differential exponential map, then $\mathsf{T}(e) : A \times A \to A \times A$ is a differential exponential map. (In fact, this makes the category of differential exponential maps a tangent category but NOT a CDC).

   Cockett, J. R. B., Cruttwell, G. S. (2014). **Differential structure, tangent structure, and SDG**. *Applied Categorical Structures, 22(2), 331-417.*

4. *Let $e : A \to A$ be a differential exponential map. Then $e \circ 0 = 0$ if and only if A is a terminal object.*

What if we had a multiplication?

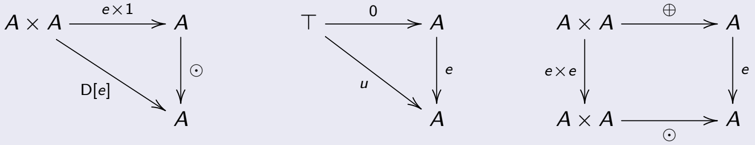
Definition

A **differential exponential semiring** is a quadruple $(A, \odot, u, e)$ consisting of an object A and maps:

$$\odot : A \times A \to A \qquad u : \top \to A \qquad e : A \to A$$

such that:

1. The differential of $\odot$ is $\mathsf{D}[\odot] = \odot \circ (\pi_0 \times \pi_1) + \odot \circ (\pi_1 \times \pi_0)$
2. $(A, \odot, u)$ is a commutative monoid (in fact $(A, \odot, u, \oplus, 0)$ is a commutative semiring);
3. The following diagrams commute:



The above three diagrams correspond to $\mathsf{D}[e^x](x, y) = e^x y$, $e^0 = 1$, and $e^{x+y} = e^x e^y$

Proposition

If $(A, \odot, u, e)$ is a differential exponential semiring, then e is a differential exponential map.

A multiplication from differential exponential maps

One can build a differential exponential semiring from a differential exponential map.

Consider the classical exponential function e^x and consider its second order directional derivative:

$$\mathsf{D}^2[e^x]((x,y),(z,w)) = e^x yz + e^x w$$

Setting $x = 0$ and $w = 0$, one obtains yz, the multiplication of real numbers!

$$\mathsf{D}^2[e^x]((0,y),(z,0)) = yz$$

The unit for this multiplication is obtain from $e^0 = 1$. Generalizing this construction allows one to show how a differential exponential map induces a differential exponential semiring.

Proposition

Let $e : A \to A$ be a differential exponential map, and define the maps $\odot_e : A \times A \to A$ and $u_e : \top \to A$ respectively as follows:

$$\odot_e := \; A \times A \xrightarrow{\langle 0,1\rangle \times \langle 1,0\rangle} (A \times A) \times (A \times A) \xrightarrow{\mathsf{D}^2[e]} A$$

$$u_e := \; \top \xrightarrow{0} A \xrightarrow{e} A$$

Then $(A, \odot_e, u_e, e)$ is a differential exponential semiring.

Proof of commutativity and unit: easy. Proof of associativity: tricky!

These constructions are inverses of each other! (so there is an isomorphism between categories)

Differential Exponential Semirings[1] – Examples in SMOOTH

Example

- For the exponential function $e^x : \mathbb{R} \to \mathbb{R}$, the induced multiplication is precisely given by the standard multiplication of real numbers, $\odot_{e^x}(x, y) = xy$ and $u_{e^x}(*) = 1$
- For the point-wise exponential function $e^x \times e^y : \mathbb{R}^2 \to \mathbb{R}^2$, we obtain the point-wise multiplication of vectors: $\odot_{e^x \times e^y}((x, y), (z, w)) = (xz, yw)$ and $u_{e^x \times e^y}(*) = (1, 1)$.
- For the complex exponential function $\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$, we obtain the multiplication:

$$\odot_\epsilon((x_1, y_1), (x_2, y_2)) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \qquad u_\epsilon(*) = (1, 0)$$

This ring structure on $\mathbb{R}^2$ is precisely that of complex numbers.

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

[1] Of course, since SMOOTH has negatives, then these examples are also rings!

Differential Exponential Semirings[2] – Examples in SMOOTH

Example

- iv For the split-complex exponential function $\epsilon' : \mathbb{R}^2 \to \mathbb{R}^2$, we obtain the multiplication:

$$\odot_\epsilon((x_1, y_1), (x_2, y_2)) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1) \qquad u_{\epsilon'}(*) = (1, 0)$$

  This ring structure on $\mathbb{R}^2$ is precisely that of split-complex numbers.

$$(x_1 + jy_1)(x_2 + ky_2) = (x_1x_2 + y_1y_2) + j(x_1y_2 + x_2y_1)$$

- v For the dual number exponential function $\mathrm{T}(e^x) : \mathbb{R}^2 \to \mathbb{R}^2$, we obtain the multiplication:

$$\odot_{\mathrm{T}(e)}((x_1, y_1), (x_2, y_2)) = (x_1x_2, x_1y_2 + y_1x_2) \qquad u_{\mathrm{T}(e)}(*) = (1, 0)$$

  This ring structure on $\mathbb{R}^2$ is precisely that of the ring of dual numbers $\mathbb{R}[\varepsilon]$.

$$(x_1 + y_1\varepsilon)(x_2 + y_2\varepsilon) = x_1x_2 + (x_1y_2 + y_1x_2)\varepsilon$$

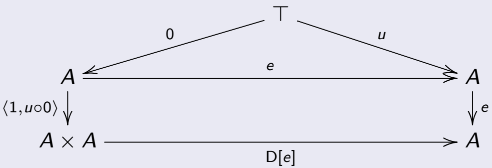
[2] Of course, since SMOOTH has negatives, then these examples are also rings!

Solution to Initial Value Problems

Every differential exponential map is also a solution to the analogue of the initial value problem $f'(x) = f(x)$ with $f(0) = 1$.

Proposition

Let $(A, \odot, u, e)$ be a differential exponential semiring. Then the following diagram commutes:



Note that the we do not say it is the unique solution! However, if one had unique solutions to differential equations, then such a solution would give a differential exponential map.

One can also use differential exponential maps to solve other differential equation: like generalizations of initial value problems of the form $f'(x) = f(x)a$ with initial condition $f(0) = b$, whose classical solution is $f(x) = e^{ax}b$.

Cockett, J. R. B., G. S. H. Cruttwell, and J-SP Lemay. "**Differential equations in a tangent category I: Complete vector fields, flows, and exponentials.**" arXiv preprint arXiv:1911.12120 (2019)

Differential Exponential Maps in the coKleisli Categories

An interesting and important source of Cartesian differential categories are coKleisli categories of differential categories. In this case a differential exponential map is of type

$$e : !A \to A$$

so a map in the coKleisli category of a comonad !

(We can give an alternative characterization as a $\otimes$-monoid morphism)

Example

Let k be a field of characteristic 0 and let VEC_k be the category of k-vector spaces and k-linear maps. VEC_k is a differential category where $!V$ is the cofree cocommutative k-coalgebra over V. If X is a basis of V then

$$!(V) \cong \bigoplus_{v \in V} k[X]$$

Clift, J. and Murfet, D., 2018. **Derivatives of Turing machines in Linear Logic**. arXiv:1805.11813.

coKl(!) is the cofree CDC over the category of k-vector spaces and *arbitrary* functions.

Garner, R, and Lemay, J-S P. "**Cartesian differential categories as skew enriched categories**." arXiv:2002.02554 (2020).

Differential exponential maps in coKl(!) correspond precisely to commutative k-algebras.

$$!(A) \longrightarrow A$$

$$p_a(x_1, \ldots, x_n) \longmapsto p(x_1, \ldots, x_n)$$

Future Work

- Find and study more examples of differential exponential maps. Places to look:
  - Cofree Cartesian Differential Categories
  - Abelian Functor Calculus
  - Categorical models of the differential λ-calculus (which in most cases has infinite sums!)

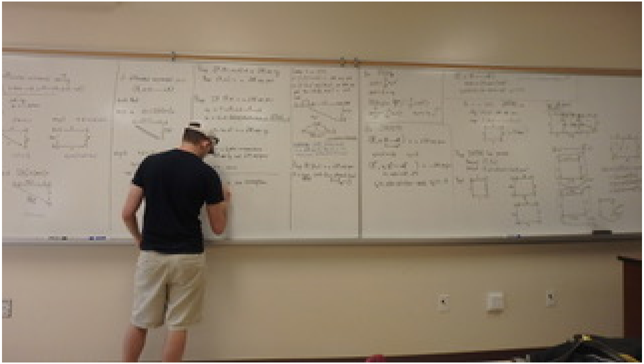
- The exponential function e^x can be defined as the inverse of the natural logarithm function $ln(x)$. Since $ln(x)$ is only partially defined, one must work in a differential restriction category to generalize the natural logarithm function in such a way that differential exponential maps arise as their (partial) inverse.

- Generalize differential exponential maps to tangent categories, and this notion should be a generalization of exponential maps for manifolds and Lie groups.

- Generalize the trigonometric functions cos and sin, and the hyperbolic functions cosh and sinh, and other kinds of functions.

- In the coKleisli category examples, many of them have a natural transformation $\mu : !!A \to !A$ which is a differential exponential map and yet looks like a monad structure on !. How close is it to actually being a monad?

Lemay, J-S. P. **Exponential Functions in Cartesian Differential Categories**. Applied Categorical Structures (2020). https://doi.org/10.1007/s10485-020-09610-0



HOPE YOU ENJOYED MY TALK!

MERCI!