Compact Hausdorff spaces and their algebraic dual

Luca Reggio

University of Oxford

Ottawa Categorical Logic Seminar, 18th March 2021

Main character: The category $\mathcal{KH}$ of compact Hausdorff spaces and continuous maps.

$\mathcal{KH}$ is extremely rich from a categorical and duality theoretic standpoint:

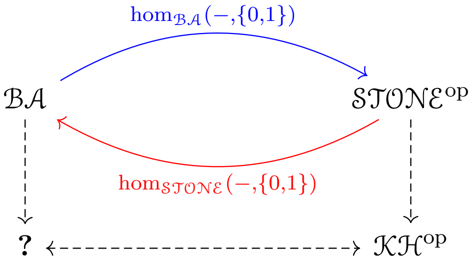
- Several (useful) dualities for $\mathcal{KH}$ have been discovered starting in the 1940s. Perhaps the best known is Gelfand & Naimark duality between $\mathcal{KH}$ and the category of commutative unital C*-algebras.
- The category $\mathcal{KH}$ itself has an algebraic nature. In fact, $\mathcal{KH}$ is monadic over $\mathcal{SET}$ (Linton '66, Manes '67).
- The fact that $\mathcal{KH}$ is categorically very well-behaved allows for an abstract/axiomatic characterisation of this category, in the spirit of Lawvere's ETCS (1964).

Plan for the talk:

1. $\mathcal{KH}^{\mathrm{op}}$: Duality theory of compact Hausdorff spaces
2. $\mathcal{KH}$: Pretopos structure and a characterisation result
3. $\mathcal{KH}_{\leq}$ and its dual: Some results and questions

$\mathcal{KH}^{\mathrm{op}}$ as an algebraic category

Dualities for $\mathcal{KH}$ can be seen as extensions of the celebrated Stone duality for Boolean algebras. Can we generalise Stone duality by removing zero-dimensionality?



$\mathcal{BA}$ = Boolean algebras
$\mathcal{STONE}$ = Zero-dimensional compact Hausdorff spaces

Classical solutions to this problem include:

- Kakutani (1941): M-spaces
- Yosida (1941): (some) abelian ℓ-groups
- Gelfand-Naimark (1943): commutative unital C*-algebras

Each of these classes contains some non-algebraic ingredient, such as the norm in C*-algebras.

Theorem (Duskin, Negrepontis (1969, 1971))

The functor $\mathcal{KH}^{\mathrm{op}} \to \mathcal{SET}$, $X \mapsto \mathcal{C}(X, [0,1])$ *is monadic. Hence,* $\mathcal{KH}^{\mathrm{op}}$ *is equivalent to a variety of algebras.*

This variety cannot be finitary, but operation symbols of at most countably infinite arity suffice to describe it.

In 1982, Isbell showed that finitely many finitary operations, along with a single operation of arity ω, are enough to describe a variety dual to $\mathcal{KH}$. The infinitary operation is:

$$\delta(f_1, f_2, f_3, \ldots) \coloneqq \sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

The problem of providing an explicit, and preferably tractable, axiomatisation of a variety dual to $\mathcal{KH}$ remained open.

In 2017, in a joint work with V. Marra, we gave a finite axiomatisation of such a variety, based on the theory of MV-algebras.

V. Marra and L. R., *Stone duality above dimension zero. Axiomatising the algebraic theory of* $\mathcal{C}(X)$, Advances in Mathematics, Vol. 307, 2017, pp. 253–287.

The standard MV-algebra is the unit interval $[0,1]$ equipped with the operation $\oplus, \neg, 0$ where

$$x \oplus y := \min(1, x+y) \quad \text{and} \quad \neg x := 1 - x.$$

The variety of MV-algebras can be defined as $\mathcal{MV} := \mathbb{HSP}([0,1])$ (assuming Chang's Completeness theorem).

Why studying the variety $\mathcal{MV}$ in the first place?

MV-algebras provide an algebraic semantics for Łukasiewicz infinite-valued propositional logic $Ł_\infty$ (1920). Thus, they stand to Łukasiewicz logic as Boolean algebras stand to classical propositional logic.

There is a functor $\hom_{\mathcal{KH}}(-, [0,1]) \colon \mathcal{KH}^{\mathrm{op}} \to \mathcal{MV}$ sending X to the set of continuous functions $\mathcal{C}(X, [0,1])$ equipped with pointwise operations.

Also, there is a functor $\hom_{\mathcal{MV}}(-, [0,1]) \colon \mathcal{MV} \to \mathcal{KH}^{\mathrm{op}}$ sending A to the set $\hom_{\mathcal{MV}}(A, [0,1])$ with the subspace topology induced by the product topology on $[0,1]^A$.

Theorem (Cignoli, Dubuc & Mundici, 2004)

$\hom_{\mathcal{MV}}(-, [0,1]) \dashv \hom_{\mathcal{KH}}(-, [0,1]) \colon \mathcal{KH}^{\mathrm{op}} \to \mathcal{MV}$. *Further, any compact Hausdorff space is fixed by this adjunction.*

So, $\mathcal{KH}$ is dual to the full subcategory of $\mathcal{MV}$ on those algebras that are fixed by this adjunction. Can we axiomatise this category?

Let us add Isbell's operation δ to the language of MV-algebras.

E.g., for all $\vec{x} = (x_1, x_2, \ldots) \in [0,1]^\omega$, $\delta(\vec{x}) := \sum_{i=1}^{\infty} \frac{x_i}{2^i}$.

More generally, for any $X \in \mathcal{KH}$, the operation δ on $\mathcal{C}(X, [0,1])$ can be interpreted pointwise.

Theorem (Marra & R, 2017)

There is a variety of algebras Δ, defined by finitely many (equational) axioms in the language $\{\delta, \oplus, \neg, 0\}$, that is dually equivalent to $\mathcal{KH}$.

- Δ is isomorphic to the full subcategory of $\mathcal{MV}$ defined by the fixed objects for the adjunction $\mathcal{KH}^{\mathrm{op}} \leftrightarrows \mathcal{MV}$.
- This yields an extension of Stone duality from $\mathcal{STONE}$ to $\mathcal{KH}$ while preserving the algebraic nature of the dual category.

Next, we sketch an application of the duality between Δ and $\mathcal{KH}$.

Recall that, for a compact Hausdorff space X, the Stone-Weierstrass theorem provides sufficient conditions for a subset $G \subseteq \mathcal{C}(X, \mathbb{R})$ to be dense in the topology induced by the uniform metric

$$\varrho(f, g) := \sup_{x \in X} \{|f(x) - g(x)|\} \quad \forall f, g \in \mathcal{C}(X, \mathbb{R}).$$

On the other hand, we can associate with Δ an equational consequence relation $\vDash_\Delta$ in a standard way. It turns out that the Stone-Weierstrass theorem is equivalent to the Beth definability property for $\vDash_\Delta$:

For any set of variables $\overline{x} \cup \{y\}$ and set of equations $\Sigma(\overline{x}, y)$, if

$$\Sigma(\overline{x}, y) \cup \Sigma(\overline{x}, z/y) \vDash_\Delta y \approx z$$

for all variables z, then there exists a term $t(\overline{x})$ such that

$$\Sigma(\overline{x}, y) \vDash_\Delta y \approx t.$$

$\mathcal{KH}$ and its pretopos structure

As we already mentioned, the category $\mathcal{KH}$ has an algebraic nature.

In his PhD thesis (1967), Manes showed that $\mathcal{KH}$ is equivalent to the category of Eilenberg-Moore algebras for the ultrafilter monad $\beta\colon \mathcal{SET} \to \mathcal{SET}$. A compact Hausdorff space X is regarded as a β-algebra by defining the structure map

$$\beta|X| \to |X|, \quad \mathcal{U} \mapsto \lim \mathcal{U}.$$

In particular, $\mathcal{KH}$ is an effective category, i.e. any (internal) equivalence relation in $\mathcal{KH}$ is the kernel pair of its coequaliser.

Effectiveness is a property typical of "algebraic categories", such as varieties of algebras, but not of categories of spaces.

E.g., $\mathcal{STONE}$ is *not* an effective category.

Another important aspect of $\mathcal{KH}$ is its structure of coherent category.

Recall that a category $\mathcal{C}$ is regular if it has finite limits and pullback-stable image factorisations.

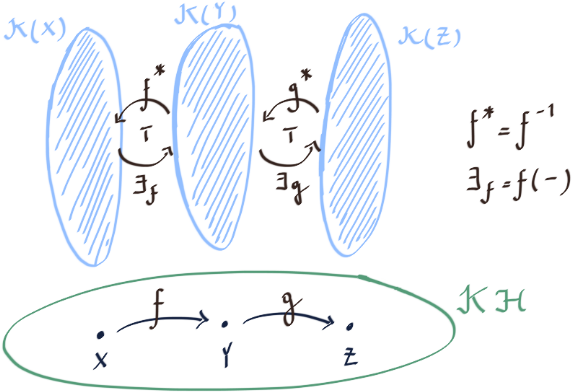
$$\mathrm{Sub}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{SL}, \quad (X \xrightarrow{f} Y) \;\mapsto\; \big(\mathrm{Sub}\, X \underset{\exists_f}{\overset{f^*}{\rightleftarrows}} \mathrm{Sub}\, Y\big), \quad \exists_f \dashv f^*$$

If, in addition, each $\mathrm{Sub}\, X$ has finite suprema and these are preserved by the pullback functors f^*, then $\mathcal{C}$ is said to be coherent.

Lemma

If $\mathcal{C}$ is coherent, then each $\mathrm{Sub}\, X$ is a distributive lattice. Hence, we get a functor $\mathrm{Sub}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{DL}$.

- $\mathcal{SET}$ is coherent.
- Any variety of algebras (regarded as a category) is regular but, in general, not coherent.
- $\mathcal{KH}$ (and also $\mathcal{STONE}$) is coherent:



A pretopos is a positive and effective coherent category.

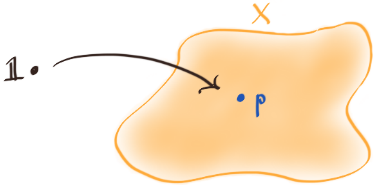
- $\mathcal{SET}$ and $\mathcal{FIN}$ are pretoposes.
- $\mathcal{KH}$ is a pretopos; $\mathcal{STONE}$ is not a pretopos.
- $\mathcal{KH}/X$ is a pretopos for every $X \in \mathcal{KH}$.

It turns out that $\mathcal{KH}$ can be characterised (up to equivalence) within the class of pretoposes.

How can we separate $\mathcal{KH}$ from $\mathcal{SET}$, $\mathcal{FIN}$, and $\mathcal{KH}/X$?

Separating $\mathcal{KH}$ and $\mathcal{KH}/X$:

Let $\mathcal{X}$ be any category admitting a terminal object $\mathbf{1}$. For any object X of $\mathcal{X}$, a point of X is a morphism $\mathbf{1} \to X$ in $\mathcal{X}$.



The category $\mathcal{X}$ is well-pointed if, for all $f, g \in \hom_{\mathcal{X}}(X, Y)$,

$$f \neq g \implies \exists p \colon \mathbf{1} \to X \text{ s.t. } f \circ p \neq g \circ p.$$

- $\mathcal{KH}$ is well-pointed.
- If $|X| \geq 2$, then $\mathcal{KH}/X$ is not well-pointed.

Separating $\mathcal{KH}$ and $\mathcal{FIN}$ is also easy. Just observe that $\mathcal{KH}$ contains all set-indexed copowers of $\mathbf{1}$, while $\mathcal{FIN}$ does not.

(For any set S, the S-fold copower of $\mathbf{1}$ in $\mathcal{KH}$ can be identified with the Stone-Čech compactification βS of the discrete space S.)

To separate $\mathcal{KH}$ and $\mathcal{SET}$, we introduce a concept of filtrality for coherent categories. This is a condition on the lattices of subobjects.

We start by recalling a useful fact (cf. the classical characterisation of Stone locales).

Lemma

Let X be a T_1-space. The following statements are equivalent:

1. *X is a Stone space.*
2. *$\mathcal{K}(X) \cong \mathcal{F}(B)$ for some Boolean algebra B.*

$\mathcal{K}(X)$ = lattice of closed subsets of X (ordered by inclusion)
$\mathcal{F}(B)$ = lattice of non-empty filters on B (ordered by reverse inclusion)

Moreover, every compact Hausdorff space is covered by a Stone space:

$$\forall X \in \mathcal{KH}\ \exists Y \in \mathcal{STONE} \text{ and a continuous surjection } Y \twoheadrightarrow X.$$

(E.g., as free compact Hausdorff spaces are Stone: $\beta|X| \twoheadrightarrow X$.)

For a bounded distributive lattice L, let $\mathcal{C}(L)$ be the Boolean center of L, and $\mathcal{F}(\mathcal{C}(L))$ the filter completion of $\mathcal{C}(L)$. There is a monotone map

$$\varphi\colon L \to \mathcal{F}(\mathcal{C}(L)), \quad x \mapsto {\uparrow} x \cap \mathcal{C}(L).$$

L is a filtral lattice if φ is an isomorphism.

An object X of a coherent category $\mathfrak{X}$ is filtral if $\operatorname{Sub} X$ is a filtral lattice. $\mathfrak{X}$ is a filtral category if every object of $\mathfrak{X}$ is covered by a filtral one; i.e., $\forall X$ there is a regular epi $Y \twoheadrightarrow X$ with Y filtral.

- The filtral objects in $\mathcal{SET}$ are precisely the finite sets. Thus, $\mathcal{SET}$ is not filtral—but $\mathcal{FIN}$ is.
- The filtral objects in $\mathcal{KH}$ are precisely the Stone spaces. Hence $\mathcal{KH}$ is filtral, and so is $\mathcal{STONE}$.

Theorem (Marra & R)

Up to equivalence, $\mathcal{KH}$ is the unique non-trivial well-pointed pretopos that is filtral and admits all set-indexed copowers of its terminal object.

For more details, see:

V. Marra and L. R., *A characterisation of the category of compact Hausdorff spaces*, Theory and Applications of Categories, Vol. 35, No. 51, 2020, pp. 1871–1906.

The conditions in the previous result are independent from each other:

	well-pointed	effective	filtral	copowers of **1**
$\mathcal{KH}/\{0,1\}$	✗	✓	✓	✓
$\mathcal{STONE}$	✓	✗	✓	✓
$\mathcal{SET}$	✓	✓	✗	✓
$\mathcal{FIN}$	✓	✓	✓	✗

Compact ordered spaces

A compact ordered space is a pair $(X, \leq)$ where

- X is a compact space;
- $\leq \subseteq X \times X$ is a partial order that is closed in $X \times X$.

E.g., $[0,1]$ with the usual Euclidean topology and linear order is a compact ordered space.

Note that every compact ordered space $(X, \leq)$ is Hausdorff because its diagonal $\leq \cap \geq$ is closed.

These spaces (and, more generally, ordered topological spaces) have been extensively studied in:

L. Nachbin, *Topology and order*, Van Nostrand Mathematical Studies, No. 4, 1965.

Let $\mathcal{KH}_{\leq}$ be the category of compact ordered spaces and continuous monotone maps, and $\mathcal{POS}$ the category of posets and monotone maps.

The forgetful functor $\mathcal{KH}_{\leq} \to \mathcal{POS}$ is monadic (Flagg, 1997).

(Open) Question: Can we characterise the category $\mathcal{KH}_{\leq}$, in the same way we characterised $\mathcal{KH}$?
(Note that $\mathcal{KH}_{\leq}$ is not balanced, hence *not* a pretopos.)

On the positive side:

Theorem (Abbadini, 2019)

The functor $\mathcal{KH}_{\leq}^{\mathrm{op}} \to \mathcal{SET}$, $X \mapsto \hom_{\mathcal{KH}_{\leq}}(X, [0,1])$ *is monadic. Hence,* $\mathcal{KH}_{\leq}^{\mathrm{op}}$ *is equivalent to a variety of algebras.*

A direct proof of the previous result was provided in:

M. Abbadini and L. R., *On the axiomatisability of the dual of compact ordered spaces*, Applied Categorical Structures, Vol. 28, 2020, pp. 921–934.

Thank you for your attention!