Lecture

or weak 2-category
or 2-category

Reminder: A <u>bicategory</u> consists of

1) a class of 0-cells $A, B, C, \ldots$
2) $\forall A, B$ 0-cells a category $\mathcal{B}(A,B)$
   the objects of $\mathcal{B}(A,B)$ are 1-cells
   the arrows of $\mathcal{B}(A,B)$ are 2-cells
3) A composition functor

$$\mathcal{B}(A,B) \times \mathcal{B}(B,C) \longrightarrow \mathcal{B}(A,C)$$

satisfying various axioms

2 key examples:

1) Rel: Objects are sets

$\mathrm{Rel}(X,Y) = \mathcal{P}(X \times Y)$, a partially ordered set (such bicategories are <u>locally ordered</u>)

2) Rng: Objects are rings

If R, S are rings

$\mathrm{Rng}(R,S)$ is the category of R-S bimodules i.e. an abelian group $(M,+)$ with an action

$$R \times M \times S \longrightarrow M$$

and maps are bimodule maps.

Goal of cartesian bicategories (Carboni-Walters) is to axiomatize Rel in the same way that toposes axiomatize the category of sets & functions.

Definition (C-W): A cartesian bicategory is a locally ordered bicategory $\mathcal{B}$ such that

1) $\mathcal{B}$ is monoidal

$\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with a special object I

satisfying monoidal axioms.

2) Every object X has a cocommutative comonoid structure

$$\Delta_X : X \to X \otimes X$$

$$t : X \to I$$

3) Every arrow is a lax comonoid homomorphism:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "R"'] \arrow[dr, phantom, "\leq"] & X \otimes X \arrow[d, "R \otimes R"] \\ Y \arrow[r, "\Delta"'] & Y \otimes Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "t_X"] \arrow[d, "R"'] \arrow[d, phantom, shift left=3, "\leq"] & \\ Y \arrow[r, "t_X"'] & I \end{tikzcd}$$

4) All the Δ and t maps have right adjoints

Ex: 1) Rel

2) Ord: 0-cells are preorders

1-cells are order ideals

$R : P \nrightarrow Q$ a subset of $R \subseteq P \times Q$

s.t.

$$x \leq_P x' \, R \, y' \leq_Q y \Rightarrow x \leq y.$$

How can I pick out the discrete objects in a cartesian bicategory?

Def (CW)

An object X in a cartesian bicategory

is <u>discrete</u> when the multiplication Δ^* and the comultiplication Δ form a <u>Frobenius algebra</u>

$$X \otimes X \xrightarrow{\Delta^*} X \xrightarrow{\Delta} X \otimes X$$

$$\shortparallel$$

$$X \otimes X \xrightarrow{1 \otimes \Delta} X \otimes X \otimes X \xrightarrow{\Delta^* \otimes 1} X \otimes X$$

A cartesian bicategory is called a <u>bicategory of relations</u> if every object is discrete.

Lemma: 1) X, Y discrete $\Rightarrow$ $X \otimes Y$ is discrete
2) I is discrete

Lemma: In Rel, every object is discrete
In Ord, the discrete objects are the equivalence relations.

Intrinsic Characterization of cartesian bicategories?

Given a bicategory which is locally ordered, one can consider the subcategory of Maps, denoted Maps $(\mathcal{B})$.

A one-cell $\overset{\wedge}{}$ is a <u>map</u>

$$f : X \to Y$$

if there is $f^* : Y \to X$ s.t.

$$id_X \leq f ; f^*$$

$$f^* ; f \leq id_Y$$

Thm: If $\mathcal{B}$ is a locally ordered cartesian bicategory,

1) Map $(\mathcal{B})$ has finite products (denoted $\times$) It's the $\otimes$.

2) Each $\mathcal{B}(X,Y)$ has finite products (denoted $\wedge$)

For any one cells,

$$F \otimes G = (p F p^*) \wedge (r G r^*)$$

where p and r are the projections.

In general, bicategorical limits & colimits are difficult. All we need is finite products

Def: Let $\mathcal{B}$ be a bicategory, and let A, B be objects. An object $A \times B$ is <u>the product of A and B</u> if equipped with 1-cells $p: A \times B \to A$ or $r: A \times B \to B$ if the canonical functor

$$\Gamma : \mathcal{B}(C, A \times B) \to \mathcal{B}(C, A) \times \mathcal{B}(C, B)$$

$$f : C \to A \times B \mapsto \langle f;p, f;r \rangle$$

is an equivalence of categories for all C.

A <u>terminal object</u> in $\mathcal{B}$ is an object 1 s.t. $\mathcal{B}(A, 1)$ is equivalent to the terminal category.

$\mathcal{B}$ has <u>finite products</u> if it has all binary products and a terminal object.

Definition (Carboni-Kelly-Walters-Wood)

A bicategory $\mathcal{B}$ is said to be <u>precartesian</u> if

1) $\mathrm{Map}(\mathcal{B})$ has finite products in the bicategorical sense

2) Each category $\mathcal{B}(X,Y)$ has finite products in the categorical sense.

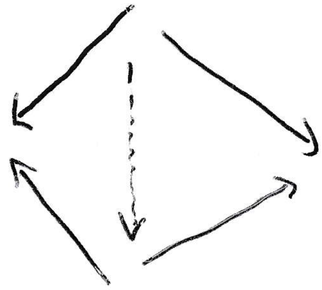
Ex: If $\mathcal{E}$ has finite limits, $\mathrm{Span}(\mathcal{E})$ is precartesian.

Objects: same as $\mathcal{E}$

1-cells: $X \nrightarrow Y$

$$\begin{tikzcd} & R \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

2-cells are maps of spans



Q: What are maps in the category of spans?

Types of (non) functors

Let $\mathcal{C}, \mathcal{D}$ be bicategories

A **lax functor** consists of $F : \mathcal{C} \to \mathcal{D}$

1) $\forall$ objects x of $\mathcal{C}$, an object $F(x)$ of $\mathcal{D}$

2) $\forall$ pair of objects x, y of $\mathcal{C}$

a functor $F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$

3) $\forall$ objects x of $\mathcal{C}$, a 2-cell

$$\alpha : \mathrm{id}_{Fx} \to F_{x,x}(\mathrm{id}_x)$$

4) For all 0-cells $x, y, z \in \mathcal{C}$
For all 1-cells $f: x \to y$ and $g: y \to z$
in $\mathcal{C}$
a 2-cell

$$\beta : F_{x,y}(f) ; F_{y,z}(g) \longrightarrow F_{x,z}(f;g)$$

These 2-cells must satisfy various equations.

A lax functor is a <u>pseudofunctor</u>
if the 2-cells α and β are invertible

Now we can use the same formula as
in the locally posetal case to define
a lax functor

$$R : X \longrightarrow A \qquad S : Y \to B$$

we define

$$R \otimes S : X \otimes Y \to A \otimes B$$

$$R \otimes S : X \otimes Y \longrightarrow A \otimes B$$

$$= (p_{X,Y} ; R ; p^*_{A,B}) \wedge (r_{X,Y} ; S ; r^*_{A,B})$$

What does all this mean?

1) $\cancel{R \otimes S}$ $X \otimes Y = X \times Y$

This exists because $Map(\mathcal{B})$ has products

2)

$$X \otimes Y \xrightarrow{p_{X,Y}} X \xrightarrow{R} A \xrightarrow{p^*_{A,B}} A \otimes B$$

$$\wedge$$

$$X \otimes Y \xrightarrow{r_{X,Y}} Y \xrightarrow{S} B \xrightarrow{r^*_{A,B}} A \otimes B$$

This exists because $\mathcal{B}(X \otimes Y, A \otimes B)$ has finite products

p, r are the projections from $Map(\mathcal{B})$

Since p, r are maps, they have adjoints p^*, r^*.

Thm: (CKWW) If $\mathcal{B}$ is a precartesian bicategory, the above construction determines lax functors

$$\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \qquad I : 1 \to \mathcal{B}$$

Note: Lots of calculations go into this.

Defn: A precartesian bicategory $\mathcal{B}$ is said to be <u>cartesian</u> if the above lax functors are pseudofunctors.

Thm (CKWW)

If $\mathcal{B}$ is a cartesian bicategory, the above data determine a symmetric monoidal bicategory.

Further results

1) Eva Aleiferi goes through the case of modules over monads, but only gets a partial result

2) Lack, Walters & Wood characterize bicategories of spans as cartesian bicategories with certain extra structure.