# Hopf Algebras and Linear Logic

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**Abstract**

It has recently become evident that categories of representations of *Hopf algebras* provide fundamental examples of monoidal categories. In this expository paper, we examine such categories as models of (multiplicative) linear logic. By varying the Hopf algebra, it is possible to model several variants of linear logic. We present models of the original commutative logic, the noncommutative logic of Lambek and Abrusci, the braided variant of the author, and the cyclic logic of Yetter. Hopf algebras provide a unifying framework for the analysis of these variants.

While these categories are monoidal closed, they lack sufficient structure to model the involutive negation of classical linear logic. We recall work of Lefschetz and Barr, in which vector spaces are endowed with an additional topological structure, called *linear topology*. The resulting category has a large class of reflexive objects, which form a $*$-autonomous category, and so model the involutive negation. We show that the monoidal closed structure of the category of representations of a Hopf algebra can be extended to this topological category in a natural and simple manner. The models we obtain have the advantage of being nondegenerate in the sense that the two multiplicative connectives, tensor and par, are not equated.

It has been recently shown by Barr that this category of topological vector spaces can be viewed as a subcategory of a certain Chu category. In an Appendix, Barr uses this equivalence to analyze the structure of its tensor product.

## 1 Introduction

Linear logic was introduced by Jean-Yves Girard in [24] as a resource sensitive logic. The general notion of categorical model of linear logic was later defined by Seely in [48]. A model is defined to be a monoidal category with certain additional structure, some of which we will discuss below. It is clear from [48] and [14] that linear logic is a logic of monoidal closed categories in much the same way that intuitionistic logic is a logic of cartesian closed categories. See [36] for an exposition of the correspondence between intuitionistic logic and cartesian closed categories.

Monoidal categories were introduced in [12] and [38] as an axiomatization of categories arising in algebraic topology. As such, categories of vector spaces are in some sense the prototypical monoidal categories. Thus it makes sense to look for models of linear logic in categories of vector spaces, possibly equipped with extra structure. Indeed, linear logic derives its name from linear algebra.

It has recently become evident that categories of representations of Hopf algebras provide fundamental examples of monoidal categories of vector spaces. This is suggested by work such as the various *Tannaka-Krein theorems*, where it is shown that a monoidal category equipped with certain additional structure is equivalent to the category of representations of a Hopf algebra. Such theorems are presented in [45], [17], [50] and [40].

It is the purpose of this paper to introduce Hopf algebras as a unifying framework for modeling several variants of multiplicative linear logic. By varying the Hopf algebra, we are able to model the traditional commutative logic, the noncommutative logic of [35] and [3], the braided variant of [15] and the cyclic linear logic of [52]. We believe it is important to have a single vehicle for modeling all these different variants, as this will allow direct comparison of the various theories. We will see that the structure of the variant we are modeling is reflected in the structure we require of the Hopf algebra. The particular Hopf algebra we choose will control the degree of symmetry of the model.

We also address the question of how to model the involutive negation of classical linear logic. This involution requires that we find categories of vector spaces which are $*$-*autonomous.* One possibility is to restrict to finite-dimensional spaces. While we do obtain models of linear logic in this way, the approach is unsatisfactory in the sense that the two multiplicative connectives, tensor and par, are equated.

Instead, we choose to use a construction of Lefschetz [37] which was subsequently examined in a categorical setting by Barr in [5]. The idea is to construct infinite dimensional reflexive spaces by adding a topological structure known as *linear topology.* By taking the dual space to be the space of linear, continuous maps, we obtain a large class of infinite-dimensional spaces which are isomorphic to their second dual space. In [5], Barr shows that the category of all such spaces, denoted $\mathcal{RTVEC}$, is $*$-autonomous, and so models the classical negation. To model the theories discussed above, we will consider continuous representations of Hopf algebras in objects in this category. The traditional work on representations of Hopf algebras is easily seen to extend to this setting.

The models so obtained are nondegenerate in the sense that the multiplicative connectives are no longer equated. While the models we obtain are nondegenerate, they do model the $MIX$ rule [20]. This is discusssed in section 6.

Hopf algebras arise in many areas of physics, computer science and combinatorics. In quantum physics, Hopf algebras arise in the study of inverse scattering; and in statistical mechanics, they are related to exactly solvable models [41]. In computer science, Hopf algebras are of interest in modelling concurrent processes [13]. In combinatorics, Hopf algebras arise as the incidence algebras of [28] and [46]. We believe that this work will allow one to relate linear logic, and its braided and noncommutative variants, to diverse areas.

We wish to emphasize at this point that this work is primarily expository in nature. While the representations of Hopf algebras in $\mathcal{RTVEC}$ had not been previously considered, this represents a straightforward generalization of the work of others. Much of the representation theory of noncocommutative Hopf algebras is worked out in [41] and [50]. An overview of some of this material is contained in [30].

It has been recently shown that the category of reflexive spaces described above is equivalent to a category of *Chu spaces* [11]. In an Appendix by Michael Barr, this equivalence is used to demonstrate that the forgetful functor from the above category of topological vector spaces to the category of vector spaces is tensor preserving.

### 1.1 Acknowledgements

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## 2 Vector Spaces and Duality

### 2.1 Categories with Duality

We briefly review the duality properties of $\mathcal{VEC}$ , the category of vector spaces and linear maps.

$\mathcal{VEC}$ has a canonical notion of duality, given by:

$$A^\perp = A \multimap k$$

Here $A \multimap B$ refers to the vector space of all linear maps from $A$ to $B$. The notion of duality we are looking for is that of a $*$-autonomous category, [10]. It was first observed by Seely [48] that these categories form the basic ingredient of models of linear logic [24].

**Definition 2.1** *A category $\mathcal{C}$ is* $*$-autonomous *if it satisfies the following:*

1. *$\mathcal{C}$ is autonomous; that is, $\mathcal{C}$ has a symmetric tensor product $A \otimes B$ and for all $A$ and $B$, an object $A \multimap B$, the* internal $HOM$, *which is adjoint to the tensor in the second variable:*

$$Hom(A \otimes B, C) \cong Hom(B, A \multimap C)$$

2. *$\mathcal{C}$ has a dualizing object $\perp$; that is, the functor $(\ )^\perp : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ defined by $A^\perp = A \multimap \perp$ is an involution, meaning that the canonical morphism:*

$$A \longrightarrow (A \multimap \perp) \multimap \perp$$

*is an isomorphism. (In an arbitrary autonomous category, any object for which this map is an isomorphism is called reflexive.)*

It is straightforward to see that a $*$-autonomous category has an internal $HOM$ of the form:

$$A \multimap B \cong (A \otimes B^\perp)^\perp$$

Or in linear logic terminology:

$$A \multimap B \cong A^\perp \parr B$$

A coherence theorem for $*$-autonomous categories was first established in the unit-free case by the author in [14]. The result made use of proof nets, a natural deduction system for multiplicative linear logic. A stronger result was then established in [16].

We also give a related definition.

**Definition 2.2** *A* compact closed category *is a symmetric monoidal category such that for each object $A$ there exists a dual object $A^\perp$, and canonical morphisms:*

$$\nu: I \to A \otimes A^\perp$$
$$\psi: A^\perp \otimes A \to I$$

*such that the usual adjunction triangles commute. ($I$ is the tensor unit.)*

It is easy to see that the operation $(-)^\perp$ can be extended to a functor, and that a compact closed category is in fact closed, with:

$$A \multimap B \cong A^\perp \otimes B$$

Note that in a compact category, the unit for tensor acts as a dualizing object.

Equivalently, a compact category could be defined as a $*$-autonomous category, with $I$ chosen as dualizing object and with the additional isomorphism:

$$- \otimes_{LT} V \ (A \otimes B)^\perp \cong A^\perp \otimes B^\perp$$

via the canonical morphism $A^\perp \otimes B^\perp \longrightarrow (A \otimes B)^\perp$.

Compact closed categories were defined by Kelly in [31]. Their coherence was studied by Kelly and La Plaza in [32], and by the author in [14].

From the point of view of linear logic, compact categories equate the two multiplicative connectives, as a consequence of the above equation. This is because the linear connectives satisfy the DeMorgan duality [24]:

$$(A \otimes B)^\perp \cong A^\perp \parr B^\perp$$

Thus, to obtain satisfactory models of the multiplicative fragment, we should construct $*$-autonomous categories which are not compact. It is the case that many naturally occurring dualities in mathematics yield compact closed categories. We will present a duality which is not compact.

## 2.2 Duality for Vector Spaces

To what extent is $\mathcal{VEC}$ a $*$-autonomous category? We have the following well-known theorem. See for example [22].

**Theorem 2.3** *A vector space $A$ is reflexive, with the base field $k$ chosen as dualizing object, if and only if $A$ is finite dimensional.*

Let $\mathcal{VEC}_{fd}$ denote the full subcategory of $\mathcal{VEC}$ consisting of finite dimensional vector spaces.

**Corollary 2.4** *$\mathcal{VEC}_{fd}$ is a $*$-autonomous category.*

Thus, $\mathcal{VEC}_{fd}$ provides a model of multiplicative (and additive) linear logic.

However, $\mathcal{VEC}_{fd}$ is a compact category. The morphism

$$\nu: k \to A \otimes A^\perp$$

is obtained by choosing a basis for $A$, say $\{e_i\}$, and then $\nu(1) = \Sigma e_i \otimes \hat{e}_i$, and then extending linearly. Here $\hat{e}_i$ denotes the dual basis element.

We now give a construction of Barr [5], based on work of Lefschetz [37], which addresses this issue. The basic idea is to recover the equation $A^{\perp\perp} \cong A$ for certain infinite dimensional vector spaces *via* topology. That is, place a topology on each vector space, and then define $A \multimap_{LT} B$ to be the linear *continuous* maps. We will thus obtain a larger category of reflexive objects, which will no longer be compact.

This idea of adding a topological structure to reduce the size of the dual space was subsequently used by Scott [47] in the theory of *continuous lattices* in order to solve recursive domain equations.

**Definition 2.5** *Let $A$ be a vector space. A topology, $\tau$, on $A$ is* linear *if it satisfies the following three properties:*

- *Addition and scalar multiplication are continuous, when the field $k$ is given the discrete topology.*
- *$\tau$ is hausdorff* [1].
- *$0 \in A$ has a neighborhood basis of open linear subspaces.*

*Let $\mathcal{TVEC}$ denote the category whose objects are vector spaces equipped with linear topologies, and whose maps are linear continuous morphisms.*

The first two requirements are standard, and say that we have a *topological vector space* [51]. The third condition is quite stringent. It implies for example that the only linear topology on a finite dimensional space is the discrete topology. Note also that the condition is much stronger than the notion of local convexity [22]. The condition will be just what we need to establish the desired duality.

An obvious example of an object of $\mathcal{TVEC}$ is an arbitrary vector space endowed with the discrete topology. We will see below that $\mathcal{TVEC}$ is a symmetric monoidal closed category, thus we have constructors for building new objects. We also have the following straightforward result:

**Theorem 2.6** *$\mathcal{TVEC}$ is complete.*

The vector space $A \multimap_{LT} B$ of linear continuous maps is endowed with the topology of pointwise convergence, *i.e.* as a subspace of the cartesian product $B^A$. Given this, the tensor product can be endowed with a linear topology to obtain an autonomous category.

**Theorem 2.7** *Given $V$ in $\mathcal{TVEC}$, the functor $V \multimap_{LT} -$ has a left adjoint, denoted $- \otimes_{LT} V$.*

**Proof**. This follows from the special adjoint functor theorem. The base field $k$ is a cogenerator. □

[1] It is actually sufficient to assume that the topology is $T_0$ in the above definition, since the $T_0$ separation axiom implies the hausdorff property for topological groups [51].

**Corollary 2.8** *(Barr)* $\mathcal{TVEC}$ *is an autonomous category.*

**Corollary 2.9** *The map* $\mu\colon A \to A^{\perp\perp}$ *is continuous, where* $A^{\perp}$ *is now defined to be* $A \multimap_{LT} k$.

Using a result due to Barr, presented in the Appendix to this paper, one may show the following. For a discussion of the Chu construction, see [9].

**Theorem 2.10** *(Barr) The underlying vector space of* $V \otimes_{LT} W$ *is* $V \otimes W$. *In other words, the forgetful functor* $\mathcal{TVEC} \to \mathcal{VEC}$ *is tensor preserving.*

**Proof**. In [11], Barr shows that $\mathcal{RTVEC}$ is equivalent to $\mathrm{Chu}_{se}(\mathcal{VEC}, k)$. (See the Appendix for a definition.) In the Appendix to this paper, Barr demonstrates that the subcategory $\mathrm{Chu}_{se}(\mathcal{VEC}, k)$ is closed under the tensor product of $\mathrm{Chu}(\mathcal{VEC}, k)$. The result follows immediately. □

**Theorem 2.11** *(Lefschetz) The map* $\mu\colon A \to A^{\perp\perp}$ *is a bijection, for all* $A$.

This amounts to showing that every element of the second dual space is given by evaluation at an element of the base space. The proof is contained in [37]. It requires some elementary results involving uniform structures on spaces which can be found in [51]. We will first need a lemma.

**Lemma 2.12** *The object* $k$ *is injective with respect to subspace inclusions in* $\mathcal{TVEC}$.

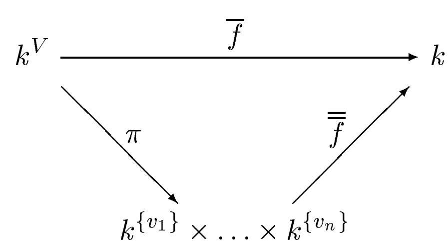
**Proof**. Suppose we have a subspace inclusion, $V \hookrightarrow W$ and suppose there is a map $\varphi\colon V \to k$. Since an object in $\mathcal{TVEC}$ is automatically a topological group, it follows that there is a canonical uniform structure on each object. See [51, p. 244]. By some standard results on uniformities, [51, p.269], the function $\varphi$ may be extended to the closure of $V$.

Thus, we may suppose $V$ is closed. The kernel of $\varphi$ is closed in $V$, and in $W$. It follows that $V/\ker\varphi \to W/\ker\varphi$ is continuous. Since $V/\ker\varphi$ is linearly compact [5, 51], it has the subspace topology with respect to $W/\ker\varphi$. Thus, it is sufficient to suppose that $V$ is one dimensional, i.e. that it is of the form $\{rv \mid r \in k\}$ for some $v \in W$. In this case, it is trivial to extend $\varphi$ to $W$. □

**Proof**. We now prove theorem 2.11. The fact that we are assuming that the space is hausdorff implies that the map is injective.

To show that $\mu$ is surjective, we must show that any map $f\colon V^{\perp} \to k$ is evaluation at an element of $V$.

Now, $V^{\perp}$ is topologized as a subspace of $k^V$. Because $k$ is injective with respect to inclusions, any map $f : V^{\perp} \to k$ extends to a map $\overline{f}\colon k^V \to k$. Because $k$ is discrete, the kernel of $\overline{f}$ is open. Any open subset of a product is the inverse image of a set under the projection to a finite product. It follows that $\overline{f}$ factors as:




Since $\overline{\overline{f}}$ has a finite dimensional vector space as its domain, and $\mu$ is bijective for finite dimensional spaces by Theorem 2.3, then it is an evaluation, for an element of the form:

$$a_1v_1 + \ldots + a_nv_n$$

It follows that $f$ is also evaluation at this element. $\square$

As already remarked, in the case of finite dimensional vector spaces, a linear topology is automatically discrete. So $\mathcal{TVEC}$ has all of the reflexive objects of $\mathcal{VEC}$. However, as we will see, $\mathcal{TVEC}$ has additional reflexive objects.

In fact, Barr gives a precise characterization of the reflexive spaces in $\mathcal{TVEC}$. We state the result, even though we will not require it.

**Theorem 2.13** *A space is reflexive if and only if every discrete linear subspace is finite dimensional.*

**Theorem 2.14** *(Barr) $\mathcal{RTVEC}$, the full subcategory of reflexive objects in $\mathcal{TVEC}$, is a $*$-autonomous category.*

First, we need several lemmas.

**Lemma 2.15** *For any $V$, $V^\perp$ is reflexive.*

**Proof**. By dualizing the map $\mu$, we get a map:

$$\mu_V^\perp : V^{\perp\perp\perp} \to V^\perp$$

In any autonomous category, this is right inverse to:

$$\mu_{V^\perp} : V^\perp \to V^{\perp\perp\perp}$$

But, since $\mu$ is bijective it is an inverse. $\square$

**Lemma 2.16** *For any $V$, and any reflexive $W$, the space $V \multimap_{LT} W$ is reflexive.*

**Proof**. Consider:

$$V \multimap_{LT} W \cong V \multimap_{LT} W^{\perp\perp} \cong (V \otimes_{LT} W^\perp) \multimap_{LT} k \cong (V \otimes_{LT} W^\perp)^\perp$$

Now use the preceding lemma. $\square$

Notice that we have shown that the internal $HOM$ is of the form $(V \otimes_{LT} W^\perp)^\perp$. It was this equation that led Barr to the definition of $*$-autonomous category.

**Lemma 2.17** *The inclusion $\mathcal{RTVEC} \to \mathcal{TVEC}$ has a left adjoint $(-)^{\perp\perp} : \mathcal{TVEC} \to \mathcal{RTVEC}$.*

**Proof**. We wish to establish the adjunction:

$$\mathcal{RTVEC}(A^{\perp\perp}, B) \cong \mathcal{TVEC}(A, B)$$

where $B$ is reflexive, and $A$ is arbitrary.

Given $f : A \to B$, take $f^{\perp\perp} : A^{\perp\perp} \to B^{\perp\perp} \cong B$.

Conversely, given $g : A^{\perp\perp} \to B$, compose with $\mu : A \to A^{\perp\perp}$ to get $\mu g : A \to B$. $\square$

**Proof**. We now prove theorem 2.14. Basically, all that is left to check is that $\mathcal{RTVEC}$ is autonomous. But, it follows from the preceding lemma that the functor $(- \otimes_{LT} V)^{\perp\perp}$ is left adjoint to $V \multimap_{LT} -$. The existence of the appropriate canonical morphisms is straightforward. The fact that we are always working in concrete categories of vector spaces guarantees that the categories we discuss in this paper automatically satisfy the coherence conditions. $\square$

**Corollary 2.18** *$\mathcal{RTVEC}$ is complete and cocomplete.*

**Proof**. As a reflective subcategory of a complete category, $\mathcal{RTVEC}$ is complete. Since a $*$-autonomous category is self-dual, it is also cocomplete. $\square$

The reason for considering this category was that we were interested in constructing nondegenerate models of linear logic, so we now state:

**Lemma 2.19** *$\mathcal{TVEC}$ is not a compact category.*

**Proof**. In order to be a compact category, $\mathcal{TVEC}$ would have to have a morphism, $\nu: I \longrightarrow A \otimes A^{\perp}$, such that the composite:

$$A \xrightarrow{\cong} I \otimes A \xrightarrow{\nu \otimes id} (A \otimes A^{\perp}) \otimes A \xrightarrow{\cong} A \otimes (A^{\perp} \otimes A) \xrightarrow{id \otimes \psi} A \otimes I \xrightarrow{\cong} A$$

is equal to the identity. Let $A$ be an infinite dimensional reflexive space, and choose a basis for $A$, $\{e_i\}$, and a basis for $A^{\perp}$, $\{f_j\}$. The image of $1 \in k$ under the map $\nu$ must be a finite sum of the form $\Sigma_{m,n} r_{m,n} e_m \otimes f_n$.

Evidently, the set $\{e_m\}$ only spans a finite dimensional subspace of $A$. If we choose $v \in A$ not contained in this subspace, it is clear that the above composite cannot equal the identity. $\square$

This argument is closely related to the fact that one can define a notion of *trace* of an endomorphism for any compact closed category [32].

# 3 Hopf Algebras and Representations

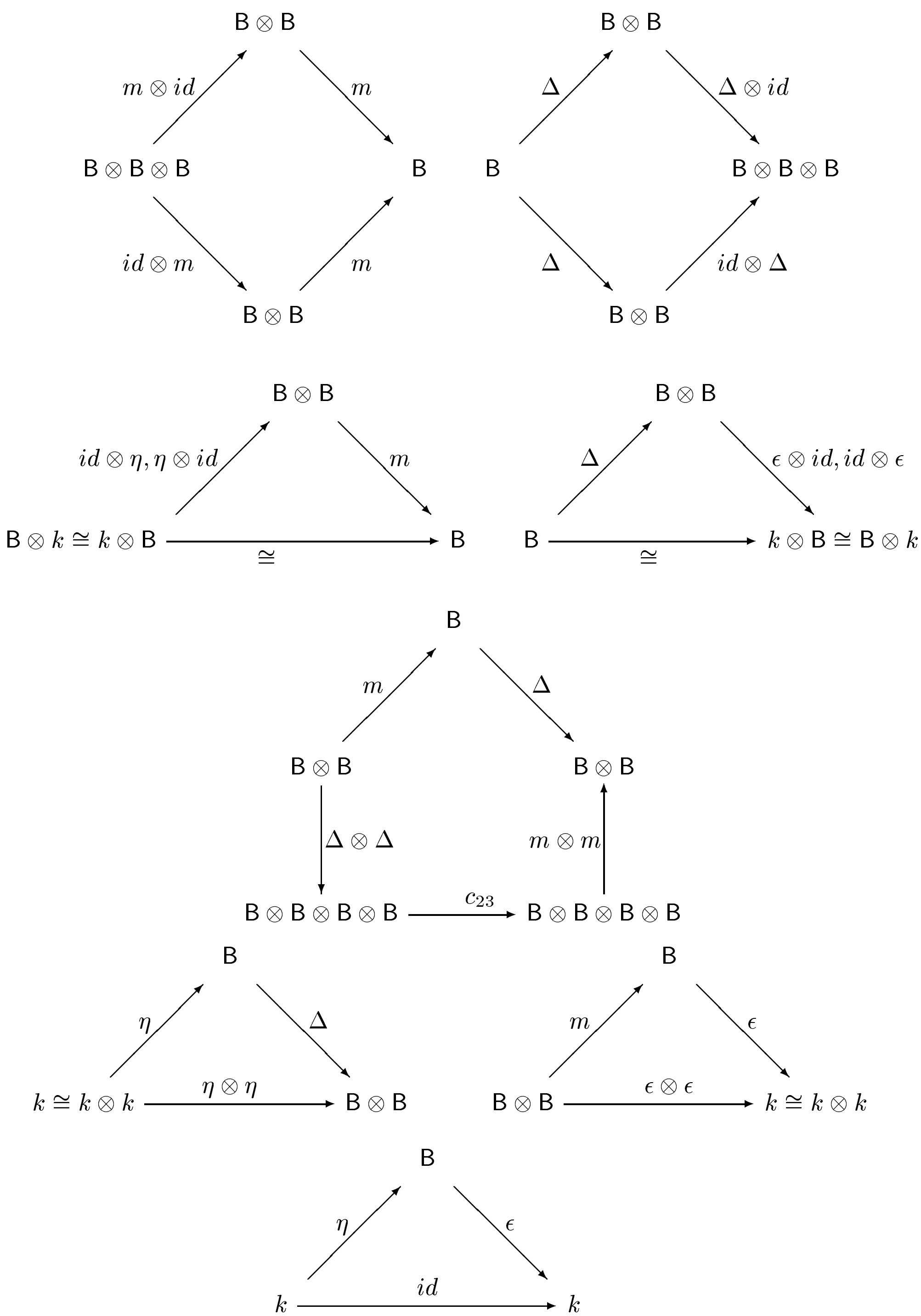
## 3.1 Bialgebras

The models we are interested in are obtained by considering representations of Hopf algebras in objects in $\mathcal{RTVEC}$. We first give the basic definitions of bialgebras and Hopf algebras. See [1] and [49] for a more complete discussion.

**Definition 3.1** *A* bialgebra, $\mathsf{B}$, *is a vector space equipped with morphisms of the following form:*

$$m: \mathsf{B} \otimes \mathsf{B} \to \mathsf{B}$$
$$\eta: k \to \mathsf{B}$$
$$\Delta: \mathsf{B} \to \mathsf{B} \otimes \mathsf{B}$$
$$\epsilon: \mathsf{B} \to k$$

*such that the following diagrams commute:*




*What these diagrams say is that we have an algebra, a coalgebra, that $\Delta$ and $\epsilon$ are maps of algebras. Note that this is equivalent to $\mu$ and $\eta$ being coalgebra maps. See [1].*

*Here, $c_{ij}$ is the canonical permutation induced by the symmetric monoidal structure.*

### 3.2 Incidence Algebras

A fundamental source of bialgebras and Hopf algebras is the theory of *incidence algebras*. These have been studied for example in [28]. They were also examined by Benson as a framework for analysis of concurrent computation, where they are refered to as *shuffle algebras* [13]. One of the traditional methods of presenting a concurrent process is to present it sequentially by interleaving the various strands. See for example the presentation of game semantics for linear logic in [2]. This operation is frequently referred to as *fair merge*. This interleaving process naturally carries the structure of a bialgebra, to be described below.

The key idea in such a presentation is the *shuffle*. Given two finite sequences, $\alpha = x_1 x_2 \ldots x_n$ and $\beta = y_1 y_2 \ldots y_m$, a shuffle is defined to be a permutation of the list $x_1, x_2, \ldots, x_n, y_1, \ldots, y_m$, such that the internal order of $\alpha$ and $\beta$ is maintained in the result. Let $\mathcal{SH}(\alpha, \beta)$ denote the set of all shuffles of $\alpha$ and $\beta$.

Following [28], we will begin with a set, $E$, then take the free monoid generated by $E$, $E^*$, and finally describe a bialgebra structure on $k[E^*]$, the vector space generated by the elements of $E^*$.

In the following, $r, s, t$ will denote elements of $k$, and $\alpha, \beta, \gamma$ will denote elements of $E^*$. The first bialgebra structure, $\mathcal{B}_1$, will have the usual monoid multiplication:

$$m_1 \left( \sum_\alpha r\alpha \otimes \sum_\beta s\beta \right) = \sum_{\alpha,\beta} rs\alpha\beta$$

The unit for this multiplication is also as usual. A corresponding comultiplication is defined as follows. Begin by defining the *deal* of a word as follows:

$$\mathcal{D}(\alpha) = \{(\beta, \gamma) | \alpha \in \mathcal{SH}(\beta, \gamma)\}$$

Then, define:

$$\Delta_1 \left( \sum_\alpha s\alpha \right) = \sum_\alpha s \left( \sum_{(\beta,\gamma) \in \mathcal{D}(\alpha)} \beta \otimes \gamma \right)$$

A counit is obtained by:

$$\epsilon_1(1_{E^*}) = 1_k$$
$$\epsilon_1(m) = 0 \text{ for all other basis elements.}$$

Now extend $\epsilon_1$ linearly.

One can now verify that $\mathcal{B}_1$ is a bialgebra. The idea here is that the word is being decomposed into all of the possible different binary concurrent processes from which it could have been constructed. By iterating $\Delta$, one obtains all finite such concurrent processes.

To define $\mathcal{B}_2$, a second bialgebra structure on $k[E^*]$, we reverse the roles of *shuffle* and *deal*. Shuffle takes the role of multiplication. Iterated applications of this multiplication take an $n$'ary concurrent process, and produce all possible interweavings into a single sequential process. Define:

$$m_2 \left( \sum_\alpha r\alpha \otimes \sum_\beta s\beta \right) = \sum_{\alpha,\beta} rs \left( \sum_{\gamma \in \mathcal{SH}(\alpha,\beta)} \gamma \right)$$

To define the coalgebra structure, first define the *cut* of a word by:

$$\Gamma(\alpha) = \sum_{\beta\gamma=\alpha} \beta \otimes \gamma$$

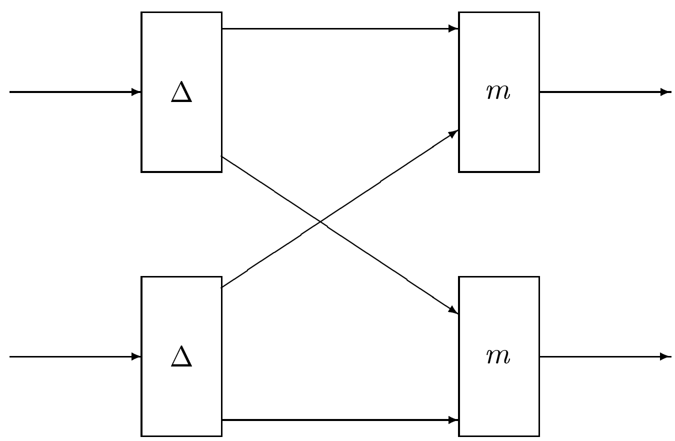
This is the sum of all decompositions of $\alpha$ into a prefix and suffix. Now define:

$$\Delta_2 \left( \sum_{\alpha} s\alpha \right) = \sum_{\alpha} s\Gamma(\alpha)$$

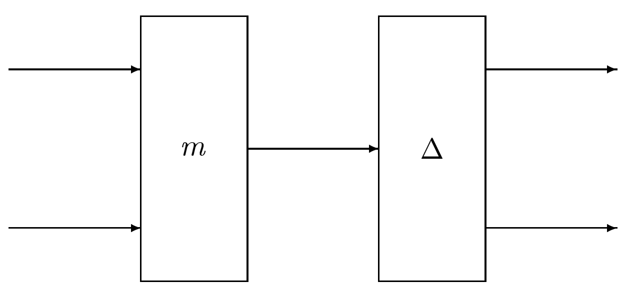
The unit and counit for $\mathcal{B}_2$ are as for $\mathcal{B}_1$.

Again, this defines a bialgebra structure.

One can represent these bialgebras with dataflow diagrams as follows. Imagine the comultiplication as being a "black box" with a single input and two outputs, as we did in the above picture. Analogously, the multiplications have two inputs and a single output. Then the bialgebra axiom $m \otimes m \circ c_{23} \circ \Delta \otimes \Delta = \Delta \circ m$ (i.e. the pentagonal diagram) can be drawn as follows:




This diagram should be equal to:



Notice the similarity to the exponential reduction rule of [27].

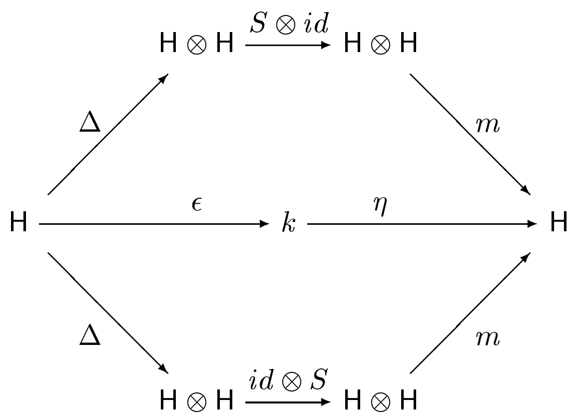
For a particularly simple example of an incidence algebra, take the set $E$ to be a singleton, say $E = x$. Then $k[E^*]$ corresponds to the polynomial ring $k[x]$, which has bialgebra structure given by:

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

with the usual algebra structure. Its antipode (see next section) is given by $S(x) = -x$.

### 3.3 Hopf Algebras

**Definition 3.2** *A vector space* $\mathsf{H}$ *is a* Hopf algebra *if it is a bialgebra, equipped with a map* $S : \mathsf{H} \to \mathsf{H}$, *called the antipode, such that:*



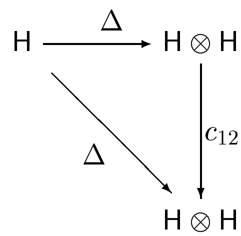

**Lemma 3.3** *If an antipode exists, it is unique, and reverses multiplication, and comultiplication. In other words, there is a map of bialgebras:*

$$(\mathsf{H}, m, \Delta) \to (\mathsf{H}, m^{op}, \Delta^{op})$$

*where* $m^{op} = mc_{12}, \Delta^{op} = c_{12}\Delta$.

**Proof**. See [1]. □

**Definition 3.4** *A Hopf algebra or bialgebra* H *is said to be* cocommutative *if:*




A *commutative* Hopf algebra is defined in terms of $m$, i.e. $m \circ c_{12} = m$.

**Example** The standard example of a Hopf algebra is the vector space, $k[G]$, generated by the elements of a group $G$. Its structure maps are given as follows:

$$\begin{gathered} m(g_1 \otimes g_2) = g_1 g_2 \\ \eta(1_k) = 1_G \\ \Delta(g) = g \otimes g \\ \epsilon(g) = 1_k \\ S(g) = g^{-1} \end{gathered}$$

Note this example is always cocommutative, but is only commutative if $G$ is abelian.

**Example** Both of the incidence bialgebras can be extended to Hopf algebras, as follows. Let $w$ be a word in $E^*$. Define $\mid w \mid$ to be the length of the word. In particular, $\mid 1_{E^*} \mid = 0$. Then define $\overline{w}$ to be the reverse word. In both $\mathcal{B}_1$ and $\mathcal{B}_2$, an antipode is defined by the equation:

$$S(w) = (-1)^{\mid w \mid} \overline{w}$$

See [46] for a discussion of this and related antipodes.

Note that $\mathcal{B}_1$ is cocommutative, while $\mathcal{B}_2$ is not. Numerous other examples are contained in [1] and [49]. We state some well-known properties of Hopf algebras. The proofs are contained in [1].

**Theorem 3.5** *The following holds for a Hopf algebra,* H, *with antipode* $S$.

- $c_{12} \circ (S \otimes S) \circ \Delta = \Delta \circ S$
- *If* H *is commutative or cocommutative,* $SS = id$.
- *If* H *is finite dimensional as a vector space, then* $S$ *is a bijection.*
- $\epsilon S = \epsilon, S\eta = \eta$

## 3.4 Modules and Representations

We now review the representation theory of Hopf algebras. The standard notion of representation is that of a *module* or *comodule*. We choose to work with modules, but virtually everything we say dualizes to the comodule case. A more general discussion of representation theory, in particular regarding the contragredient representation discussed below, is contained in [21].

**Definition 3.6** *Given a bialgebra or Hopf algebra* $\mathsf{H}$*, a* module *over* $\mathsf{H}$ *is a vector space* $V$*, equipped with a* $k$*-linear map, called an* $\mathsf{H}$*-action* $\rho\colon \mathsf{H}\otimes V \to V$ *such that:*

$$\begin{array}{ccc} \mathsf{H}\otimes\mathsf{H}\otimes V & \xrightarrow{id\otimes\rho} & \mathsf{H}\otimes V \\ \Big\downarrow{\scriptstyle m\otimes id} & & \Big\downarrow{\scriptstyle \rho} \\ \mathsf{H}\otimes V & \xrightarrow{\rho} & V \end{array} \qquad \begin{array}{ccc} V & \xleftarrow{\rho} & \mathsf{H}\otimes V \\ & \nwarrow{\scriptstyle \cong} & \Big\uparrow{\scriptstyle \eta\otimes id} \\ & & k\otimes V \end{array}$$

*If* $(V,\rho)$ *and* $(W,\tau)$ *are modules, then a map of modules, sometimes called an* $\mathsf{H}$*-map, is a* $k$*-linear map* $f\colon V\to W$ *such that:*

$$\begin{array}{ccc} \mathsf{H}\otimes V & \xrightarrow{id\otimes f} & \mathsf{H}\otimes W \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \tau} \\ V & \xrightarrow{f} & W \end{array}$$

*We thus obtain a category* $\mathcal{MOD}(\mathsf{H})$.

This definition is a straightforward generalization of the notion of *group representation.* If $U$ and $V$ are modules, then $U\otimes V$ has a natural module structure given by:

$$\mathsf{H}\otimes U\otimes V \xrightarrow{\Delta\otimes id} \mathsf{H}\otimes\mathsf{H}\otimes U\otimes V \xrightarrow{c_{23}} \mathsf{H}\otimes U\otimes\mathsf{H}\otimes V \xrightarrow{\rho\otimes\rho} U\otimes V$$

Denote this module as $U\otimes_{\mathsf{H}} V$.

**Theorem 3.7** *If* $\mathsf{H}$ *is a bialgebra or Hopf algebra,* $\mathcal{MOD}(\mathsf{H})$ *is a monoidal category.*

**Note 3.8** *The unit for tensor is given by the field* $k$ *with module structure induced by* $\epsilon$*.*

**Theorem 3.9** *If* $\mathsf{H}$ *is cocommutative,* $\mathcal{MOD}(\mathsf{H})$ *is symmetric and the forgetful functor to the category of vector spaces is a symmetric monoidal functor.*

There are slightly weaker conditions than cocommutativity under which the category can be symmetric, by a map other than the usual symmetry. This is related to the *triangular* elements of [41]. But, in general, the category of modules for a noncocommutative Hopf algebra is not symmetric. The intuition is that we will use the Hopf algebra structure to control the degree of symmetry of the model.

### 3.5 $\mathcal{MOD}(\mathsf{H})$ as an Autonomous Category

We now explore the closed structure of $\mathcal{MOD}(\mathsf{H})$, where $\mathsf{H}$ is a cocommutative Hopf algebra. We have already observed that $\mathcal{MOD}(\mathsf{H})$ has a tensor product, and that the underlying space of the tensor is the tensor of the underlying spaces. The same will apply to the internal $HOM$. Given two modules, $A$ and $B$, we define a module $A \multimap_{\mathsf{H}} B$. Its underlying space will be $A \multimap_k B$, the space of $k$-linear maps. Before defining its module structure, we will need some notation. The standard notation for Hopf algebras is developed in [49].

**Notation** If $h, h' \in \mathsf{H}$, and $a \in A$, denote:

$$\Delta(h) = \Sigma(h_1 \otimes h_2)$$
$$m(h \otimes h') = hh'$$
$$\rho(h \otimes a) = ha$$

Furthermore, since $\mathsf{H}$ is coassociative, we can unambiguously refer to:

$$(\Delta \otimes id)\Delta(h) = (id \otimes \Delta)\Delta(h) = \Sigma h_1 \otimes h_2 \otimes h_3$$

and so on. Given $f : A \to B$, $h \in \mathsf{H}$, and $a \in A$, we define the action of $h$ on $f$ by:

$$(hf)(a) = \Sigma h_1 f(S(h_2)a)$$

In the case of the Hopf algebra $k[G]$, this formula amounts to the standard contragredient representation:

$$(gf)(a) = gf(g^{-1}a)$$

At this point, we observe the following standard result. See for example [9].

**Theorem 3.10** *With the above structure, $\mathcal{MOD}(\mathsf{H})$ becomes an autonomous category. Furthermore, the forgetful functor to $\mathcal{VEC}$ is an autonomous functor.*

**Proof**. The first part of this theorem will follow from a more general result later in the paper, theorem 4.4. The second part is obvious. □

The fact that the forgetful functor is autonomous implies that many statements about $\mathcal{VEC}$ remain true when considering $\mathcal{MOD}(\mathsf{H})$. For example:

**Corollary 3.11** *$\mathcal{MOD}_{fd}(\mathsf{H})$, the full subcategory of finite dimensional modules, is a compact category.*

Thus, we have the same problem in modeling linear logic. But again, Barr's topological construction resolves this difficulty, as we will see below. The fact that the forgetful functor to $\mathcal{VEC}$ preserves the autonomous structure implies that the topological construction of the previous section applies here as well.

### 3.6 Topological Modules

We will define a category $\mathcal{TMOD}(\mathsf{H})$, of modules over $\mathsf{H}$, endowed with linear topologies, and again obtain a model for which tensor and par are no longer equated.

**Definition 3.12** *The category $\mathcal{TMOD}(\mathsf{H})$ is defined as follows. Objects are modules $(A, \rho)$ such that $A$ is equipped with a linear topology, and such that the action of $\mathsf{H}$ on $A$ is continuous, when $\mathsf{H}$ is given the discrete topology. Maps are $\mathsf{H}$-maps which are also continuous.*

It now remains to study reflexive objects in $\mathcal{TMOD}(\mathsf{H})$. But, all of the previous work extends in a straightforward way at this level of generality.

In particular, the map $\mu: A \to A^{\perp\perp}$ exists and is again a bijection. Thus we conclude:

**Theorem 3.13** *Let $\mathsf{H}$ be a cocommutative Hopf algebra. $\mathcal{RTMOD}(\mathsf{H})$, the full subcategory of reflexive objects in $\mathcal{TMOD}(\mathsf{H})$, is a reflective subcategory of $\mathcal{TMOD}(\mathsf{H})$ via the functor $(-)^{\perp\perp}$. Furthermore, $\mathcal{RTMOD}(\mathsf{H})$ is a $*$-autonomous category, and the forgetful functor to the category $\mathcal{RTVEC}$ is a $*$-autonomous functor.*

**Corollary 3.14** *$\mathcal{RTMOD}(\mathsf{H})$ is complete and cocomplete.*

We have thus constructed another example of a $*$-autonomous category which is not compact. These models should be of some interest for several reasons. First, they validate the $MIX$ rule, as will be discussed below. Also, by altering the required structure of the Hopf algebra, i.e. cocommutative vs. quasitriangular vs. noncocommutative, we alter the structure of the resulting category of modules. (We have only stated the theorem in the cocommutative case, but the result goes through at this level of generality. See below.) Thus Hopf algebras give access to models of a wide variety of variants of linear logic. The above construction can be viewed as a uniform method of constructing models of a number of variants of linear logic.

## 4 Biautonomous categories

Our definition of autonomous category is that it be symmetric, monoidal and closed. If we drop the requirement that the tensor be symmetric, then one should consider categories with two internal $HOM$'s. Thus we should have adjunctions of the form:

$$HOM(A \otimes B, C) \cong HOM(B, A \multimap C)$$
$$HOM(A \otimes B, C) \cong HOM(A, C \mathbin{\circ\!-} B)$$

This is the definition of *biautonomous* category, an obvious generalization of the symmetric case. Of course, if the tensor happens to be symmetric, this will induce an isomorphism between the two $HOM$'s.

The earliest coherence results for biautonomous categories were obtained by Lambek in [34]. This is where the notion of categories as deductive systems was introduced.

Analogously, to define a nonsymmetric analogue of categories with dualizing objects one needs two duals, $A^\perp$ and ${}^\perp A$. (The dualizing object for each will be the same.)
These will be subject to the isomorphisms:

$$^{\perp}(A^{\perp}) \cong (^{\perp}A)^{\perp} \cong A$$

More specifically, a biautonomous category, has a canonical morphism:

$$A \to\ ^{\perp}(A^{\perp}) \cong (^{\perp}A)^{\perp}$$

and if this map is an isomorphism, then we have a *bi-$*$-autonomous* category. (In general, there will be no relationship between $A$ and $A^{\perp\perp}$ in the nonsymmetric case.) Furthermore, if the two dual functors distribute over tensor, then we obtain the noncommutative version of compact category.

We now discuss a variant of these categories.

**Definition 4.1** *If in a bi-$*$-autonomous category, the dualizing object, $\perp$, has the property that:*

$$^{\perp}A \cong A^{\perp}$$

*or equivalently:*

$$A \multimap \perp \cong \perp \circ\!\!- A$$

*then $\perp$ is said to be* cyclic. *A $*$-autonomous category with such a dualizing object is also said to be cyclic.*

In the posetal case, these are the Girard quantales, and were introduced by Yetter in [52] and studied by Rosenthal in [44]. A notion of proof net for this theory is contained in [52].

Yetter's *cyclic linear logic* is obtained by replacing the usual exchange rule with:

$$\frac{\vdash A_1, A_2, \ldots, A_n}{\vdash A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}}$$

where $\sigma$ is a *cyclic* element of the symmetric group on $n$ letters. It is straightforward to verify that a $*$-autonomous category with a cyclic dualizing object validates this rule.

### 4.1 $\mathcal{MOD}(\mathsf{H})$ as a Biautonomous Category

A good candidate for a nontrivial biautonomous category is the category of modules over a non-cocommutative Hopf algebra. However, it turns out that an additional assumption is needed. The extra assumption necessary is that the antipode be bijective. In this case, we will define two module structures on the space of $k$-linear maps between modules. We first state the following well-known elementary result about Hopf algebras.

**Lemma 4.2** *Let $\mathsf{H}$ be an arbitrary Hopf algebra. Define a bialgebra $\mathsf{H}'$, the* opposite bialgebra *by reversing the multiplication of $\mathsf{H}$ (but not the comultiplication). Then, $\mathsf{H}'$ has an antipode if and only if $S$ is bijective, in which case the antipode for $\mathsf{H}'$ is given by $S^{-1}$.*

**Definition 4.3** *Given an arbitrary Hopf algebra $\mathsf{H}$ with bijective antipode, and two $\mathsf{H}$-modules, $A$ and $B$, we will define two new $\mathsf{H}$-modules, $A \multimap B$ and $B \circ\!\!- A$, as follows. In both cases, the underlying space will be $A \multimap_k B$, the space of $k$-linear maps.*

*The action of $B \circ\!\!- A$ will be the previously defined action, that is:*

$$(hf)(a) = \Sigma h_1 f(S(h_2)a)$$

*The action of $A \multimap B$ is defined by:*

$$(hf)(a) = \Sigma h_2 f(S^{-1}(h_1)a)$$

The following is proved in [41].

**Theorem 4.4** *Let* $\mathsf{H}$ *be a Hopf algebra with bijective antipode. Then with the actions defined above,* $\mathcal{MOD}(\mathsf{H})$ *is a biautonomous category. The adjoint relation:*

$$HOM(A \otimes B, C) \cong HOM(B, A \multimap C)$$

*holds whether or not the antipode is bijective. In the case of a cocommutative Hopf algebra, the two internal* $HOM$*'s are equal.*

**Proof**. One need only verify that the usual evaluation and coevaluation maps commute with the $\mathsf{H}$-module structure. For completeness, we present one such calculation. Here $Lev$ is evaluation for the left internal $HOM$, $f \in B \circ\!\!- A$, and $a \in A$.

$$\begin{aligned} Lev(h(f \otimes a)) &= \Sigma Lev(h_1 f \otimes h_2 a) \\ &= \Sigma Lev(h_1 f(Sh_2(\_)) \otimes h_3 a) \\ &= \Sigma h_1 f(Sh_2 h_3 a) \\ &= \Sigma h_1 f(\eta\epsilon(h_2)a) \\ &= \Sigma h_1 f(\epsilon(h_2)a) \\ &= \Sigma h_1 \epsilon(h_2) f(a) \\ &= hf(a) \\ &= hLev(f \otimes a) \end{aligned}$$

The other three cases are handled similarly. Note in particular that:

$$\Sigma S^{-1}(h_2)h_1 = \Sigma h_2 S^{-1} h_1 = \eta\epsilon(h)$$

because $S^{-1}$ is the antipode in the opposite bialgebra, as noted above.

In the cocommutative case, remember that $SS = id$. □

# 5 The Braided Case

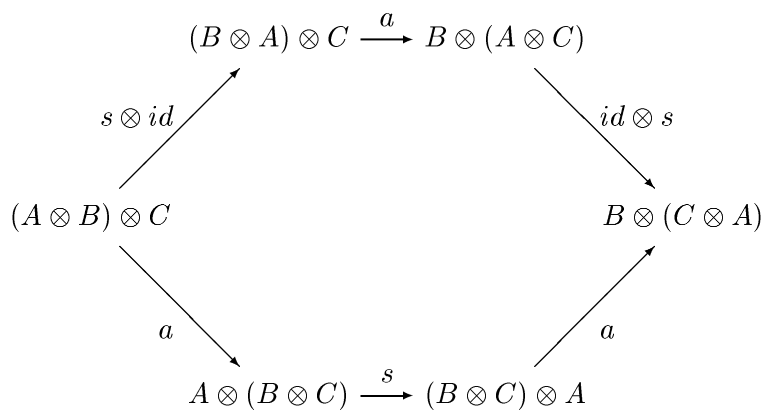
## 5.1 Braided Monoidal and Braided Closed Categories

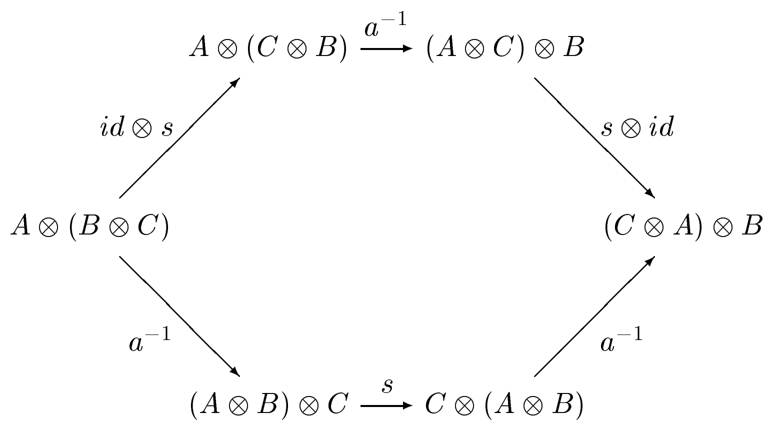
Braided monoidal categories were introduced by Joyal and Street in [29]. They are a generalization of symmetric monoidal categories in that only certain of the equations of a symmetry are satisfied.

**Definition 5.1** *A* braided monoidal category *is a monoidal category equipped with a natural isomorphism:*

$$s = s_{AB} : A \otimes B \to B \otimes A$$

*such that the following two diagrams commute (note that the a's are the associativity isomorphisms):*




**Note 5.2** *To obtain symmetric monoidal categories, one adds the additional requirement that* $s^2 = id$*. In this case,* $s$ *is instead denoted by* $c$*, as is traditional.*

Braided monoidal categories derive their name from the following theorem, see [29] and [30].

**Theorem 5.3** *(Joyal-Street) The category of braids is the free braided monoidal category on one object. This category has the natural numbers for objects, and braids on natural numbers for morphisms, with the usual braid composition. Tensor product is given by braid juxtaposition.*

Thus, braids supply the notion of *Kelly-Mac Lane graph* for this theory, see [33] and [14]. In [14], the author showed that morphisms in an autonomous or $*$-autonomous category can be represented by proof nets, and that the Kelly-Mac Lane graph can be recovered by tracing paths through the net. With this in mind, we introduced a notion of *braided linear logic* and *braided proof net* in [15]. These are used to study the coherence of braided closed categories.

## 5.2 Quasitriangular Hopf Algebras

*Quasitriangular* Hopf algebras, or *quantum groups* were introduced by Drinfel'd in [18]. The definition is a weakening of the cocommutativity condition discussed previously, which is too strong to correctly model quantum phenomena. An overview of the physical meaning is contained in [39] and [41]. The resulting category of modules is only "quasisymmetric" in the terminology of Majid. But, following the work of Joyal and Street, quasisymmetry amounts to a braiding.

If we drop the cocommutativity condition on Hopf algebras, then the symmetry map on underlying vector spaces:

$$c(x \otimes y) = y \otimes x$$

is no longer an $\mathsf{H}$-map in general. A quasitriangular Hopf algebra contains a distinguished invertible element $\mathcal{R} \in \mathsf{H} \otimes \mathsf{H}$ such that the map $s : A \otimes B \to B \otimes A$, defined by:

$$s(x \otimes y) = \mathcal{R}(y \otimes x)$$

(where $\mathsf{H} \otimes \mathsf{H}$ acts on $B \otimes A$ in the evident way) induces a braiding on $\mathcal{MOD}(\mathsf{H})$.

First, this map, $s$, must be an $\mathsf{H}$-map. This means that:

$$h\mathcal{R}(y \otimes x) = \mathcal{R}c_{21}(h(x \otimes y))$$

Or, equivalently (remember that $h$ acts on a tensor product *via* the comultiplication):

$$\Delta(h)\mathcal{R} = \mathcal{R}c_{21}(\Delta(h))$$

This can be written as:

$$\mathcal{R}\Delta^{op}\mathcal{R}^{-1} = \Delta$$

For $s$ to be a braiding, we must further have the following two equations, which correspond to the two commutative diagrams in the definition of braided monoidal category. Let $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$, then we must have:

$$(\Delta \otimes 1_{\mathsf{H}})(\mathcal{R}) = (\sum 1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2)(\sum \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2)$$
$$(1_{\mathsf{H}} \otimes \Delta)(\mathcal{R}) = (\sum \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes 1)(\sum \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2)$$

These equations can be interpreted as saying that $\mathsf{H}$ is cocommutative "up to conjugation by $\mathcal{R}$". We now have the following result.

**Theorem 5.4** *If $\mathsf{H}$ is a quasitriangular Hopf algebra, then $\mathcal{MOD}(\mathsf{H})$ is a braided monoidal category.*

Quasitriangular Hopf algebras exist in abundance. For example, any cocommutative Hopf algebra is quasitriangular, trivially. Just take $\mathcal{R} = 1_{\mathsf{H}} \otimes 1_{\mathsf{H}}$.

Two constructions are of particular interest. Majid, in [39], studies the *quantum double*, first defined by Drinfel'd, [18]. This construction assigns to every finite dimensional Hopf algebra another finite dimensional Hopf algebra with a canonical quasitriangular element. Majid shows that it is an example of a *double crossproduct*, a generalization of semidirect products.

The other work is that of Gerstenhaber, Giaquinto and Schack, [23]. They construct quantum groups *via* algebraic deformation theory. Thus certain Hopf algebras can be continuously deformed to obtain quasitriangular Hopf algebras. The cohomology theory of bialgebras and Hopf algebras is crucial to this study.

### 5.3 General Theorem

We sum up with the following result, which establishes the connection to the various fragments of linear logic. The proof is evident from the previous discussion.

**Theorem 5.5** *Let* $\mathsf{H}$ *be a Hopf algebra with bijective antipode. Then* $\mathcal{RTMOD}(\mathsf{H})$ *is a bi-$*$-autonomous category. If* $\mathsf{H}$ *is quasitriangular,* $\mathcal{RTMOD}(\mathsf{H})$ *is braided.*

It is shown in [42] that a quasitriangular Hopf algebra necessarily has a bijective antipode.

### 5.4 Cyclic Linear Logic

**Theorem 5.6** *Let* $\mathsf{H}$ *be a Hopf algebra with involutive antipode, i.e.* $SS = id$*. Then* $\mathcal{RTMOD}(\mathsf{H})$ *is a bi-$*$-autonomous category which also models Yetter's cyclic rule.*

**Proof**. One need only check that $A^{\perp} = {}^{\perp}A$. This is a consequence of the following calculation, where $f : A \to k$, and $x \in A$:

$$\begin{aligned} \Sigma h_1 f(S(h_2)x) &= \Sigma\epsilon(h_1)f(S(h_2)x) \\ &= \Sigma f(S(\epsilon(h_1)h_2)x) \\ &= f(S(h)x) \end{aligned}$$

and similarly for the other action. □

We believe that this theorem illustrates the great utility and flexibility of Hopf algebraic models of linear logic. It is possible to model a great many theories *via* the same basic construction, and the properties of the theory we are modeling are reflected in the algebraic structure.

## 6 The $MIX$ Rule

The $MIX$ rule was first mentioned by Girard in [24]. It was studied extensively by Fleury and Rétoré in [20], where they develop a notion of proof net for this theory. To obtain proof nets for this theory, one simply replaces the usual condition that the subgraph only be acyclic and connected for each switch setting to simply require that each subgraph be acyclic. The $MIX$ rule also has a natural computational interpretation, as discussed in [20]. It has arisen in the context of the Abramsky-Jagadeesan game semantics, [2], where they obtain a strong completeness theorem for the theory mLL+$MIX$. It was also used by the author in obtaining a general coherence theorem for monoidal closed categories, [14]. The theory mLL+$MIX$ is the correct level of generality for deriving such a theory. We now describe this rule and its categorical semantics.

Syntactically, (using Girard's one sided sequents), the rule is defined as follows.

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

One method of modeling the MIX rule in a $*$-autonomus category is to require an isomorphism between the two multiplicative units, i.e. $\top \cong \bot$.

If we call this new biunit $I$, then one recovers the $MIX$ rule *via* the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\Gamma^{\perp} \vdash \qquad \vdash \Delta}}{\Gamma^{\perp} \vdash \mathbf{I} \qquad \mathbf{I} \vdash \Delta}}{\Gamma^{\perp} \vdash \Delta}}{\vdash \Gamma, \Delta}$$

Most models of linear logic have this property. It is true for example of coherence spaces [24] and the game semantics of [2]. It is evident that all of the models discussed above have this property, as well.

**Theorem 6.1** *All of the categories described above validate the MIX rule. We thus obtain models of the theory mLL+MIX.*

# 7 Conclusion

As we already mentioned, Barr's construction works with a great degree of generality. Similar constructions work in the category of Banach spaces with contractions as maps, see [6]. Thus, nontrivial models can be obtained here as well. An advantage of using normed spaces is that in these categories, the two additive connectives will be distinguished as well.

Another possibility is to change the topology on function spaces as in [7],[8]. One approach is to use the topology of uniform convergence, rather than the pointwise topology. A comparison of these sorts of models, in particular the computational meaning of the two possible topologies should be worthwhile.

It was suggested earlier in the paper that the specified structure of the Hopf algebra, for example the quasitriangular element, corresponded to the structural rules of the fragment one is trying to model. This leads to the idea that a Hopf algebra or bialgebra can be viewed as an abstract notion of syntax for the specification of a deductive system. It should be possible to formalize this idea. In this interpretation, the category of modules would serve as a canonical semantics. From this viewpoint, an important class of theorems are the *Tannaka-Krein* theorems. Such theorems say that one can recover an algebraic structure from its category of representations. The original Tannaka-Krein theorem was for the case of compact groups. This was generalized to Hopf algebras in [45], [17] and [50]. Another such theorem was obtained by S. Majid for *quasihopf algebras*, see [40]. A general discussion is contained in [30]. From a logical viewpoint, these can be interpreted as completeness theorems, or more specifically conceptual completeness theorems, such as those obtained in [43]. Tannakian categories [17] provide an axiomatic framework for such dualities, and a suitable generalization may be appropriate for linear logic.

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**Appendix-Separability of tensor in Chu categories of vector spaces**

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# A Introduction

If $\mathcal{V}$ is a symmetric monoidal closed category with pullbacks and $\perp$ is an object of $\mathcal{V}$, there is a construction described in [Chu, 1979], of a $*$-autonomous category based on $\mathcal{V}$. Very briefly, an object of $\mathrm{Chu}(\mathcal{V},\perp)$ is a pair $(V,V')$ of objects of $\mathcal{V}$ together with a pairing $\langle -,-\rangle : V \otimes V' \longrightarrow \perp$. These objects have been called *Chu spaces* because in concrete examples one can think of $V'$ as being a kind of topology on $V$ ([Pratt, 1993]).

An object $(V,V')$ is called *separated* if the induced map $V \longrightarrow V' \multimap \perp$ is monic and *extensional* if the induced $V' \longrightarrow V \multimap \perp$ is monic. It is not true in general that the tensor product of two separated Chu spaces is separated nor that the internal hom of two extensional spaces is extensional. The

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purpose of this note is to show that these claims (which are equivalent to each other) are true when the ground category is the category of vector spaces.

A complete introduction to Chu spaces and $*$-autonomous categories can be found in [Barr, 1991]. See also [Barr, to appear]. We give a quick sketch.

## A.1 The Chu construction

A morphism $(f, f') : (V, V') \longrightarrow (W, W')$ is a pair of arrows $f : V \longrightarrow W$ and $f' : W' \longrightarrow V'$ such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes W' & \xrightarrow{f \otimes W'} & W \otimes W' \\ {\scriptstyle V \otimes f'}\downarrow & & \downarrow{\scriptstyle \langle -,- \rangle} \\ V \otimes V' & \xrightarrow[\langle -,- \rangle]{} & \bot \end{array}$$

In terms of elements, this says that for all $v \in V$ and $w' \in W'$, $\langle fv, w' \rangle = \langle v, f'w' \rangle$.

Another way of expressing this is that the diagram

$$\begin{array}{ccc} Hom((V,V'),(W,W')) & \longrightarrow & Hom(V,W) \\ \downarrow & & \downarrow \\ Hom(W',V') & \longrightarrow & Hom(V \otimes W', \bot) \end{array} \tag{1}$$

is a pullback.

The category $\mathrm{Chu}(\mathcal{V}, \bot)$ is a $*$-autonomous category, with the following structures. First, for Chu spaces $(V, V')$ and $(W, W')$, define a Vect-valued hom $(V, V') \multimapdot (W, W')$ so that:

$$\begin{array}{ccc} (V,V') \multimapdot (W,W') & \longrightarrow & V \multimap W \\ \downarrow & & \downarrow \\ W' \multimap V' & \longrightarrow & (V \otimes W') \multimap \bot \end{array}$$

is a pullback. Note that this diagram is simply the obvious strengthening of (1) to $\mathcal{V}$. Now we define

$$(V,V') \otimes (W,W') = (V \otimes W, (V,V') \multimapdot (W',W))$$

The pairing is most easily given in terms of elements. For $v \in V$, $w \in W$, $f : V \longrightarrow W'$ and $f' : W \longrightarrow V'$, we let $\langle v \otimes w, (f, f') \rangle = \langle w, fv \rangle = \langle v, f'w \rangle$, The internal hom is given similarly by

$$(V,V') \multimap (W,W') = ((V,V') \multimapdot (W,W'), V \otimes W')$$

The duality is $(V, V')^* = (V, V') \multimap (\bot, \top) = (V', V)$, where $\top$ is the tensor unit.

It is shown in [Barr, 1991] (where "separated" is called "left separated" and "extensional" is called "right separated") that in general the tensor product of extensional spaces is extensional, but, as shown in the following example, the tensor product of separated spaces may fail to be separated. The opposite happens for the internal hom, where the internal hom of two separated spaces is separated, but the internal hom of extensional spaces may not be extensional. We will see that this does not happen in $\mathrm{Chu}(\mathrm{Vect}, K)$. In light of the duality between the tensor and internal hom, it will follow, when we show that the tensor product of separated spaces is separated, that the internal hom of extensional spaces is extensional.

### A.2 An example: abelian groups

Consider the category $\mathrm{Chu}(\mathrm{Ab}, T)$ where $T = \mathbf{R}/\mathbf{Z}$ is the circle group. There is a Chu space $(\mathbf{Q}, \mathbf{Z})$ using any pairing $\mathbf{Q} \otimes \mathbf{Z} \cong \mathbf{Q} \to T$ that embeds $\mathbf{Q}$ into $T$. Multiplication by any irrational number will do. Such a pairing is both separated and extensional. On the other hand, $(\mathbf{Q}, \mathbf{Z}) \otimes (\mathbf{Q}, \mathbf{Z}) = (\mathbf{Q} \otimes \mathbf{Q}, (\mathbf{Q}, \mathbf{Z}) \mathrel{-\!\bullet} (\mathbf{Q}, \mathbf{Z}))$ with the latter being the pullback

$$\begin{array}{ccc} (\mathbf{Q}, \mathbf{Z}) \mathrel{-\!\bullet} (\mathbf{Q}, \mathbf{Z}) & \longrightarrow & \mathbf{Q} \multimap \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Q} \multimap \mathbf{Z} & \longrightarrow & \mathbf{Q} \otimes \mathbf{Q} \multimap T \end{array}$$

which is evidently 0 since there are no non-zero homomorphisms $\mathbf{Q} \to \mathbf{Z}$. Thus $(\mathbf{Q}, \mathbf{Z}) \otimes (\mathbf{Q}, \mathbf{Z}) = (\mathbf{Q}, 0)$, which is evidently not separated.

## B Vector spaces

Let $K$ be a field and $\mathrm{Vect} = \mathrm{Vect}_K$ be the category of vector spaces over a field.

**Theorem B.1** *In* $\mathrm{Chu}(\mathrm{Vect}_K, K)$*, the tensor product of separated spaces is separated and the internal hom of extensional spaces is extensional.*

**Proof**. This means that the map $V \otimes W \to ((V, V') \mathrel{-\!\bullet} (W', W))^*$ must be shown to be injective. If not, there are finite dimensional subspaces $V_0$ and $W_0$ of $V$ and $W$ such that the composite

$$V_0 \otimes W_0 \to V \otimes W \to ((V, V') \mathrel{-\!\bullet} (W', W))^*$$

is not injective either. We will show that this is impossible by first showing that $(V_0, V_0^*)$ is a subobject, in fact split subobject, of $(V, V')$ and similarly for $(W_0, W_0^*) \to (W, W')$ and that the upper and right arrows in

$$\begin{array}{ccc} V_0 \otimes W_0 & \longrightarrow & ((V_0, V_0^*) \mathrel{-\!\bullet} (W_0^*, W_0))^* \\ \downarrow & & \downarrow \\ V \otimes W & \longrightarrow & ((V, V^*) \mathrel{-\!\bullet} (W^*, W))^* \end{array} \tag{2}$$

are injective. Actually, the top arrow is an isomorphism.

From $V_0 \longrightarrow V$, we have $V' \longrightarrow V^* \longrightarrow V_0^*$ which gives us a morphism $(V_0, V_0^*) \longrightarrow (V, V')$. I claim that $V' \to V_0^*$ is surjective. If not, the arrow factors through a proper subspace, say $U \subsetneq V_0^*$ and then from vector space duality, we have $V_0 \to U^* \longrightarrow V'^*$. Then we have a commutative square:

$$\begin{array}{ccc} V_0 & \longrightarrow\!\!\!\!\!\rightarrow & U^* \\ \downarrow & & \downarrow \\ V & \longrightarrow & V'^* \end{array}$$

and the diagonal fill-in gives $U^* \longrightarrow V$ such that the upper triangle commutes, which implies, along with $V_0 \hookrightarrow V$, that $V_0 = U^*$, and then that $U = V_0^*$.

Next, I claim that the injection $(f, f') : (V_0, V_0^*) \hookrightarrow (V, V')$ is split. Let $v_1, \ldots, v_n$ be a basis of $V_0$ and let $v_1^*, \ldots, v_n^*$ be the dual basis of $V_0^*$. This means that $\langle v_i, v_j^* \rangle = \delta_{ij}$. Since $f'$ is surjective, let $v_1', \ldots, v_n' \in V'$ be vectors such that $f'v_i' = v_i^*$, for $i = 1, \ldots, n$. Now define $g : V \longrightarrow V_0$ by

$$g(v) = \sum_{i=1}^{n} \langle v, v_i' \rangle v_i$$

and $g' : V_0^* \longrightarrow V'$ by $g'v_i^* = v_i'$. We want to show that $(g, g') : (V, V') \longrightarrow (V_0, V_0^*)$ is a map in the category and that it splits $(f, f')$. For the first, we have, for $v \in V$ and $i = 1, \ldots, n$,

$$\langle gv, v_i^* \rangle = \langle \sum_{j=1}^{n} \langle v, v_j' \rangle v_j, v_i^* \rangle = \langle v, v_i' \rangle = \langle v, g'v_i^* \rangle$$

so that $(g, g')$ is a morphism. Then we have for $i = 1, \ldots, n$,

$$gf(v_i) = \sum_{j=1}^{n} \langle fv_i, v_j' \rangle v_i = \sum_{j=1}^{n} \langle v_i, f'v_j' \rangle v_i = \sum_{j=1}^{n} \langle v_i, v_j^* \rangle v_i = v_i$$

We could similarly show that $f'g' = \mathrm{id}$ but it is unnecessary since in the subcategory of separable extensional objects, the two halves of a map determine each other.

Now the diagram:

$$\begin{array}{ccc} (V_0, V_0^*) \,{-\!\!\bullet}\, (W_0, W_0^*) & \longrightarrow & V_0 \multimap W_0^* \\ \downarrow & & \downarrow \\ W_0 \multimap V_0^* & \longrightarrow & (V_0 \multimap W_0)^* \end{array}$$

is a pullback while the bottom and right arrows are isomorphisms and hence

$$(V_0, V_0^*) \,{-\!\!\bullet}\, (W_0^*, W_0) = (V_0 \multimap W_0)^*$$

so that

$$((V_0, V_0^*) \multimap\!\bullet\, (W_0^*, W_0))^* = V_0 \multimap W_0$$

which shows that the top map in (2) is an isomorphism, as claimed. As for the right hand map in (2), we observe that $(V_0, V_0^*) \longrightarrow (V, V')$ is split monic and, dually, $(W', W) \longrightarrow (W_0^*, W_0)$ is split epic so that

$$(V, V') \multimap\!\bullet\, (W', W) \longrightarrow (V_0, V_0^*) \multimap\!\bullet\, (W_0^*, W_0)$$

is split epic and then

$$((V_0, V_0^*) \multimap\!\bullet\, (W_0^*, W_0))^* \longrightarrow ((V, V') \multimap\!\bullet\, (W', W))^*$$

is (split) monic, as required.

□

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