Derivations in Codifferential Categories

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Dedication. The authors dedicate this work to the memory of Jim Lambek.

Abstract

Derivations provide a way of transporting ideas from the calculus of manifolds to algebraic settings where there is no sensible notion of limit. In this paper, we consider derivations in certain monoidal categories, called *codifferential categories*. Differential categories were introduced as the categorical framework for modelling differential linear logic. The deriving transform of a differential category, which models the differentiation inference rule, is a derivation in the dual category. We here explore that derivation's universality.

One of the key structures associated to a codifferential category is an *algebra modality*. This is a monad T such that each object of the form TC is canonically an associative, commutative algebra. Consequently, every T-algebra has a canonical commutative algebra structure, and we show that universal derivations for these algebras can be constructed quite generally.

It is a standard result that there is a bijection between derivations from an associative algebra A to an A-module M and algebra homomorphisms over A from A to $A \oplus M$, with $A \oplus M$ being considered as an infinitesimal extension of A. We lift this correspondence to our setting by showing that in a codifferential category there is a canonical T-algebra structure on $A \oplus M$. We call T-algebra morphisms from TA to this T-algebra structure *Beck T-derivations*. This yields a novel, generalized notion of derivation.

The remainder of the paper is devoted to exploring consequences of that definition. Along the way, we prove that the symmetric algebra construction in any suitable symmetric monoidal category provides an example of codifferential structure, and using this, we give an alternative definition for differential and codifferential categories.

1 Introduction

The theory of Kähler differentials [15, 20] provides an analogue of the theory of differential forms and all of its various uses in settings other than the usual setting of smooth manifolds. They were originally introduced by Kähler as an abstract algebraic notion of differential form. One of their advantages is that they can be applied to varieties which are not also smooth manifolds, such as singular varieties in characteristic 0 or arbitrary varieties over a field of characteristic p. In a setting where one does not have access to limits, one can still talk about *derivations*. That is to say one passes from the variety to its coordinate ring, and then considers a module over that ring. A derivation is then a linear map from the algebra to the module satisfying the Leibniz rule. The module of *Kähler differentials* or *Kähler module* is then a module equipped with a universal derivation. As usual, such a module is unique up to isomorphism.

Since this initial work, the idea of extending differential forms to more and more abstract settings has advanced in a number of different directions. As one important example, we mention the noncommutative differential forms that arise in noncommutative geometry [18].

Differential linear logic [11, 12] arose originally from semantic concerns. Ehrhard [9, 10] had constructed several models of linear logic [14] in which the hom-sets had a natural differentiation operator. Ehrhard and Regnier then described this operation as a sequent rule and represented it as a construction and a rewrite rule for both interaction nets and for λ-calculus. The corresponding categorical structures were introduced in [3, 4] and called *differential categories* and *cartesian differential categories*. Cartesian differential categories are an axiomatization of the coKleisli category of a differential category.

The notion of *Kähler category* [2] began with the observation that the *deriving transform*, the key feature of differential categories, is a derivation and, under certain assumptions, has a universal property discussed below. (Actually, we must work with the dual notion of codifferential category. If we worked with coalgebras and coderivations, we could work in differential categories and all of the following work, suitably *op*-ed, would still hold.) It thus seemed likely that an abstract monoidal setting in which Kähler differential modules could be defined would apply to differential categories. In fact, the original paper only partially resolved this issue. In the present paper, we provide a much more satisfying answer by generalizing the notion of derivation to take into account all of the codifferential structure, thereby establishing a suitable universal property in full generality.

A Kähler category is an additive, symmetric monoidal category with an *algebra modality*, i.e. a monad T such that each object of the form TC is equipped with a commutative, associative algebra structure and several coherence equations hold, and each associative algebra has an object of universal derivations. In essence, we are requiring a Kähler module for each free T-algebra.

The present paper extends the work of [2] in several ways. It is not surprising that, given all the structure at hand, one can endow every T-algebra with the structure of a commutative, associative algebra. We show that in a Kähler category, one can use the existence of Kähler objects for free T-algebras to derive Kähler objects for all algebras[1] that arise in this way. Thus if the algebra category is monadic over the base, we can derive Kähler modules for all algebras by a single uniform procedure. These results follow from the M.Sc. thesis of the third author [21].

We also tackle the idea of what it means to be a derivation. It is well-known [6] that if A is a commutative algebra and M is an A-module, then there is a canonical algebra structure on $A \oplus M$

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[1]We realize that the unavoidable use of the word *algebra* in two different ways is confusing. The word algebra without a $T-$ in front of it will always mean commutative, associative algebra.

such that derivations from A to M are in bijective correspondence to algebra maps over A from A to $A \oplus M$. Essentially the algebra $A \oplus M$ is the extension of A by M-infinitesimals. This idea was used in a much more general setting by Beck [1].

While this is a straightforward calculation, it has far-reaching generalizations. First we show that in a codifferential category, given a T-algebra (A, ν) and a module M over the algebra associated to A, there is a canonical T-algebra structure on $A \oplus M$ which under the passage from T-algebras to algebras yields the traditional associative algebra structure on $A \oplus M$ from [1]. We call this T-algebra $W(A, M)$. We then define a *Beck T-derivation on A valued in M* to be a map of T-algebras from (A, ν) to $W(A, M)$ in the slice category over A. Beck T-derivations can be equivalently given by morphisms $\partial \colon A \to M$ satisfying a *chain rule* condition with respect to T.

We show that the symmetric algebra monad yields a codifferential category in a very general setting and in this case, our notion of Beck T-derivation is equivalent to the usual notion of derivation.

We define a *module of Kähler T-differentials* to be an A-module with a universal Beck T-derivation. We then show that the deriving transform in a codifferential category is always universal in this sense. In fact, every T-algebra has a universal T-derivation. Our analysis also yields an equivalent definition of differential category we believe will be valuable in generalizations of this abstract notion of differentiation. For example, it generalizes in a straightforward way to noncommutative settings.

We note that in [8], Dubuc and Kock define a notion of derivation on an algebra of a *Fermat theory*, the latter being a finitary set-based algebraic theory extending the theory of commutative rings and satisfying a certain axiom. It would be interesting to compare their notion with the notion of T-derivation defined here in the monoidal context of codifferential categories.

The extension of Kähler categories and codifferential categories to noncommutative settings is an important project, and work of this sort has already begun [7]. In that paper, Cockett has explored the implications of demanding for each T-algebra A and each A-bimodule M a *given* T-algebra structure on $A \oplus M$ satisfying certain axioms, whereas here we have shown that in the setting of a codifferential category, a T-algebra structure on $A \oplus M$ can be defined in terms of the given codifferential structure.

2 Derivations and categorical frameworks

This section covers the theory of derivations, both in its classical formulation with respect to algebras over a field and several of its more abstract categorical formulations.

2.1 Classical case

Derivations were originally considered for associative, commutative algebras over a field and are employed in algebraic geometry and commutative algebra [13, 15].

Definition 2.1. Let k be a commutative ring, A a commutative k-algebra, and M an A-module. (All modules throughout the paper will be *left* modules.)

A *k-derivation* from A to M is a k-linear map $\partial \colon A \longrightarrow M$ such that $\partial(aa') = a\partial(a') + a'\partial(a)$.

One can readily verify under this definition that $\partial(1) = 0$ and hence $\partial(r) = 0$ for any $r \in k$.

Definition 2.2. Let A be a k-algebra. A *module of A-differential forms* is an A-module Ω_A together with a k-derivation $\partial : A \longrightarrow \Omega_A$ which is universal in the following sense: For any A-module M, and for any k-derivation $\partial' : A \longrightarrow M$, there exists a unique A-module homomorphism $f : \Omega_A \longrightarrow M$ such that $\partial' = \partial; f$.

Lemma 2.3. *For any commutative k-algebra A, a module of A-differential forms exists.*

There are several well-known constructions. The most straightforward, although the resulting description is not that useful, is obtained by constructing the free A-module generated by the symbols $\{\partial a \mid a \in A\}$ divided out by the evident relations, most significantly $\partial(aa') = a\partial(a') + a'\partial(a)$.

2.2 Derivations as algebra maps

We suppose we are working in the category of vector spaces over a field k, that A is a commutative k-algebra and M an A-module. Define an associative, commutative algebra structure on $A \oplus M$ by

$$(a, m) \cdot (a', m') = (aa', am' + a'm)$$

It is evident that this is associative, commutative and unital. We will refer to this algebra structure as the *infinitesimal extension of A by M*. But its interest comes from the following observation.

Lemma 2.4. *There is a bijective correspondence between k-derivations from A to M and k-algebra homomorphisms from A to $A \oplus M$ which are the identity in the first component. Or more succinctly:*

$$Der_k(A, M) \cong Alg/A(A, A \oplus M)$$

Here, Alg/A is the slice category of objects over A in the category Alg of k-algebras.

We also note that it is straightforward to lift this result to the level of additive symmetric monoidal categories, see Section 2.3. The notions of commutative algebra and module are expressible in any symmetric monoidal category. Once one has additive structure then the notion of derivation is definable as well. The correspondence of Lemma 2.4 then extends to this more general setting. Lemma 2.4 also provided Jon Beck [1] a starting point for a far-reaching generalization of the notion of derivation for the purposes of cohomology theory. One of the primary contributions of this paper is to lift the correspondence of Lemma 2.4 to the level of codifferential categories. The fact that these ideas continue to hold at this level is testament to the importance of Beck's ideas about cohomology.

2.3 Categorical structure

It is a standard observation [19, 17] that the notions of algebra (monoid) and module over an algebra make sense in any monoidal category and the notion of commutative algebra makes sense in any symmetric monoidal category. But to discuss derivations for an algebra we also need additive structure.

Definition 2.5.   1. A symmetric monoidal category $\mathcal{C}$ is *additive* if it is enriched over commutative monoids and the tensor functor is additive in both variables.[2].

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[2]In particular, we only need addition and unit on Hom-sets, rather than abelian group structure.

2. Let (A, m_A, e_A) be an algebra in an additive symmetric monoidal category[3], and $M = \langle M, \bullet_M : A \otimes M \longrightarrow M \rangle$ an A-module. Then a *derivation to M* is an arrow $\partial : A \longrightarrow M$ such that (with m being the multiplication)

$$m; \partial = c; 1 \otimes \partial; \bullet_M + 1 \otimes \partial; \bullet_M \qquad and \qquad \partial(1) = 0$$

Remark 2.6. We note that Lemma 2.4 holds at this level of generality as well. Indeed, given a commutative algebra A in an additive symmetric monoidal category $\mathcal{C}$ with finite coproducts (equivalently, finite biproducts) and an A-module M, we can equip $A \oplus M$ with the structure of a commutative algebra [2]. Derivations $A \to M$ then correspond to maps $A \to A \oplus M$ in the slice category Alg/A over A in the category Alg of commutative algebras in $\mathcal{C}$ [2]. As noted in [2, §4.2], every map of A-modules $h : M \to N$ determines an algebra map $1 \oplus h : A \oplus M \to A \oplus N$, whence each derivation $\partial : A \to M$ determines a composite derivation $A \xrightarrow{\partial} M \xrightarrow{h} N$. Further, given a map of commutative algebras $g : A \to B$, each B-module N determines an A-module N_A, the *restriction of scalars of N along g*, consisting of the object N of $\mathcal{C}$ equipped with the composite A-action

$$A \otimes N \xrightarrow{g \otimes 1} B \otimes N \xrightarrow{\bullet_N} B \ .$$

Moreover, given an algebra map $g : A \to B$ and a derivation $\partial : B \to N$, the composite $A \xrightarrow{g} B \xrightarrow{\partial} N$ is a derivation $A \to N_A$.

As for most algebraic structures, when one adds in an appropriate notion of universality, the result is a very powerful mathematical object. For derivations, we obtain the module of *Kähler differentials* or *Kähler module*. We cite [15, 20] for calculations and examples.

Definition 2.7. Let $\mathcal{C}$ be an additive symmetric monoidal category and let A be a commutative algebra in $\mathcal{C}$. A *module of Kähler differentials* is an A-module Ω_A together with a derivation $\partial : A \longrightarrow \Omega_A$, such that for every A-module M, and for every derivation $\partial' : A \longrightarrow M$, there exists a unique A-module map $h : \Omega_A \longrightarrow M$ such that $\partial; h = \partial'$.

$$A \xrightarrow{\ \partial\ } \Omega_A$$
$$\partial' \searrow \quad \downarrow h$$
$$M$$

An axiomatization of a very different sort which attempted to capture the process of differentiation axiomatically is the theory of *differential categories* [3]. Since in this paper we wish to work with algebras and derivations as opposed to coalgebras and coderivations, we work in the dual theory of *codifferential categories*.

Definition 2.8. An *algebra modality* on a symmetric monoidal category $\mathcal{C}$ consists of a monad (T, μ, η) on $\mathcal{C}$, and for each object C in $\mathcal{C}$, a pair of morphisms (note we are denoting the tensor unit by k)

$$m : T(C) \otimes T(C) \longrightarrow T(C), \qquad e : k \longrightarrow T(C)$$

making $T(C)$ a commutative algebra such that this family of associative algebra structures satisfies evident naturality conditions [2].

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[3]We will use the notation m_A and e_A for the multiplication and unit for A.

Definition 2.9. An additive symmetric monoidal category with an algebra modality is a *codifferential category* if it is also equipped with a *deriving transform*[4], *i.e.* a transformation natural in C

$$d_{T(C)} : T(C) \longrightarrow T(C) \otimes C$$

satisfying the following four equations[5]:

(d1) $e; d = 0$ *(Derivative of a constant is 0.)*

(d2) $m; d = (1 \otimes d); (m \otimes 1) + (d \otimes 1); c; (m \otimes 1)$ (where c is the appropriate symmetry)   *(Leibniz Rule)*

(d3) $\eta; d = e \otimes 1$ *(Derivative of a linear function is constant.)*

(d4) $\mu; d = d; \mu \otimes d; m \otimes 1$ *(Chain Rule)*

We make the following evident observation, noting that the morphism $u_C^{TC} := e \otimes 1 \colon C = k \otimes C \to T(C) \otimes C$ exhibits $T(C) \otimes C$ as the free $T(C)$-module on C.

Lemma 2.10. *When $T(C) \otimes C$ is considered as the free $T(C)$-module generated by C, then the above deriving transform is a derivation.*

This leaves the question of its universality. We know there is a universal property for the object $T(C) \otimes C$ as the free $T(C)$-module generated by C. Is this sufficient to guarantee the universality necessary to be a Kähler module? With this question in mind, the paper [2] introduced the notion of a *Kähler category* but only partially answered this question.

Definition 2.11. A *Kähler category* is an additive symmetric monoidal category with

- a monad T,

- a (commutative) algebra modality for T,

- for all objects C, a $T(C)$-module of Kähler differential forms, satisfying the universal property of a Kähler module.

Thus the previous question can be formulated as whether every codifferential category is a Kähler category. The original paper [2] had a partial answer to this question. In the present paper, we give a much more satisfying answer to this question. The key is to abstract even further the notion of derivation. We use ideas from Jon Beck's remarkable thesis [1]. This will be covered in Section 4.

2.4   Universal derivations for T-algebras

In a category with an algebra modality we may endow each T-algebra with the structure of a commutative algebra, in such a way that the structure map of the T-algebra is a morphism of associative algebras. Since universal derivations are a priori only defined for the algebras arising axiomatically in a Kähler category, it is natural to ask if universal derivations from these new associative algebras exist and, if so, how they are constructed. We examine this issue now and

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[4]We use the terminology of a *deriving transform* in both differential and codifferential categories.

[5]For simplicity, we write as if the monoidal structure is strict.

demonstrate that there is a very pleasing answer. The construction of such Kähler modules is from the third author's M.Sc. thesis [21]. We first note the following procedure for assigning algebra structure to T-algebras.

Theorem 2.12. *Let $\mathcal{C}$ be a symmetric monoidal category equipped with an algebra modality T. The following construction determines a functor from the category of T-algebras to the category of commutative associative algebras in $\mathcal{C}$. Let (A, ν) be a T-algebra in such a category. Define the multiplication for an algebra structure on A by the formula*

$$A \otimes A \xrightarrow{\eta \otimes \eta} TA \otimes TA \xrightarrow{m} TA \xrightarrow{\nu} A$$

with unit given by

$$k \xrightarrow{e} TA \xrightarrow{\nu} A$$

In particular, every map of T-algebras becomes an associative algebra map.

Also note that if we apply this construction to the free T-algebra (TA, μ), we get back the original associative algebra (TA, m, e).

Definition 2.13. Let $\mathcal{C}$ be an additive symmetric monoidal category. Let A and B be algebras with universal derivations as in the diagram below. Let $f \colon A \to B$ be an algebra homomorphism. Define $\Omega_f : \Omega_A \longrightarrow \Omega_B$ to be the unique morphism of A-modules making

$$
\begin{array}{ccc}
\Omega_A & \xrightarrow{\ \Omega_f\ } & \Omega_B \\[1em]
\Big\uparrow {\scriptstyle d_A} & & \Big\uparrow {\scriptstyle d_B} \\[1em]
A & \xrightarrow[\ f\ ]{} & B
\end{array}
$$

commute, which exists by universality of d_A. One can verify that $\Omega_{(-)}$ is functorial.

The existence of Kähler modules for free T-algebras entails that Kähler modules for arbitrary T-algebras can be obtained by taking a quotient, as is seen in the following theorem.

Theorem 2.14. *Defining $\Omega_{A,\nu}$ as the following coequalizer*

$$\Omega_{T^2 A} \overset{\Omega_\mu}{\underset{\Omega_{T\nu}}{\rightrightarrows}} \Omega_{TA} \xrightarrow{\ \Omega_\nu\ } \Omega_{A,\nu}$$

gives us the module of Kähler differentials for T-algebra (A, ν).

This result was in the M.Sc. thesis of the third author [21]. We do not give a proof of this result here as it can be obtained in a method similar to Theorem 4.23. We also note that, under suitable hypotheses, the existence of Kähler modules for arbitrary commutative algebras follows from Theorem 4.23.

3   The symmetric algebra monad

The most canonical example of an algebra modality is the symmetric algebra construction. This construction as applied to the category of vector spaces gives one of the most basic examples of a codifferential category. In this case, elements of the symmetric algebra are essentially polynomials, which are differentiated in the evident way. A similar construction works on the category of sets and relations. What we observe here is that the symmetric algebra construction provides examples of codifferential categories in a much more general setting.

First, we need to explore a theme which will be the centrepiece of the last sections of the paper. This is the idea of viewing derivations as algebra homomorphisms.

Remark 3.1. *For the remainder of this section, we assume $\mathcal{C}$ is an additive symmetric monoidal category with finite coproducts and reflexive coequalizers, the latter of which are preserved by the tensor product in each variable. Let Alg be the category of commutative algebras in $\mathcal{C}$, and suppose that the forgetful functor $Alg \longrightarrow \mathcal{C}$ has a left adjoint. The resulting adjunction is then monadic; denote its induced monad by $\mathcal{S} = (S, \eta^S, \mu^S)$, so that $Alg \cong \mathcal{C}^{\mathcal{S}}$, and we henceforth identify these categories. See [19] for details.*

3.1   Structure related to the symmetric algebra

We will also need the following straightforward observation:

Proposition 3.2. *The (commutative) algebra modalities on $\mathcal{C}$ are in bijective correspondence to pairs (T, ψ), where T is a monad and ψ is a monad morphism $\psi \colon \mathsf{S} \to T$. Such a morphism induces a functor*

$$F_\psi \colon T\text{-}Alg \to \mathsf{S}\text{-}Alg$$

Furthermore, the map $\psi_C \colon \mathsf{S}C \to TC$ is a map of algebras.

3.2   Codifferential structure

Definition 3.3. Given an object C in $\mathcal{C}$, recall that $SC \otimes C$ is the free SC-module on C. Hence by Remark 2.6, the direct sum $SC \oplus (SC \otimes C)$ carries the structure of an algebra, and derivations $SC \to (SC \otimes C)$ correspond to algebra homomorphisms $SC \to SC \oplus (SC \otimes C)$ whose first coordinate is the identity. But since SC is the free algebra on C, the latter correspond to morphisms $C \to SC \oplus (SC \otimes C)$ whose first coordinate is $\eta \colon C \to SC$.

So let $d_{SC} \colon SC \longrightarrow SC \otimes C$ be the derivation corresponding to the algebra homomorphism $SC \longrightarrow SC \oplus (SC \otimes C)$ given on generators as $\begin{pmatrix} \eta_C \\ u_C \end{pmatrix} \colon C \longrightarrow SC \oplus (SC \otimes C)$, where u_C is the map $u_C \colon C \cong k \otimes C \xrightarrow{\ e \otimes 1\ } SC \otimes C$.

Theorem 3.4. $(\mathcal{C}, S, d)$ *is a codifferential category.*

Proof. S is a commutative algebra modality on $\mathcal{C}$. Since each d_{SC} is by definition a derivation, the Leibniz rule holds and precomposing d_{SC} by e_{SC} is the zero map. By the definition of d_{SC},

$$\eta_C ; \begin{pmatrix} 1_{SC} \\ d_{SC} \end{pmatrix} = \begin{pmatrix} \eta_C \\ u_C \end{pmatrix} \colon C \longrightarrow SC \oplus (SC \otimes C)$$

so that

$$\eta_C; d_{SC} = \eta_C; \begin{pmatrix} 1_{SC} \\ d_{SC} \end{pmatrix}; \pi_2 = \begin{pmatrix} \eta_C \\ u_C \end{pmatrix}; \pi_2 = u_C$$

and consequently (d3) holds.

It remains only to demonstrate naturality of d and adherence to the chain rule condition. For naturality, consider a map $f : C \longrightarrow D$ in $\mathcal{C}$; naturality of d is equivalent to the commutativity of the following square:

$$
\begin{array}{ccc}
SC & \xrightarrow{\begin{pmatrix} 1 \\ d_{SC} \end{pmatrix}} & SC \oplus (SC \otimes C) \\
{\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Sf \oplus (Sf \otimes f)} \\
SD & \xrightarrow[\begin{pmatrix} 1 \\ d_{SD} \end{pmatrix}]{} & SD \oplus (SD \otimes D)
\end{array}
$$

Since each morphism in the square is an algebra morphism, commutativity of this square may be demonstrated by showing that the square is commutative when preceded by $\eta_C : C \longrightarrow SC$. By naturality of η and definition of d_D we have on the left:

$$\eta_C; Sf; \begin{pmatrix} 1 \\ d_{SD} \end{pmatrix} = f; \eta_D; \begin{pmatrix} 1 \\ d_{SD} \end{pmatrix} = f; \begin{pmatrix} \eta_D \\ u_D \end{pmatrix}$$

By naturality of η and u and by definition of d we have on the right:

$$\eta_C; \begin{pmatrix} 1 \\ d_{SC} \end{pmatrix}; Sf \oplus (Sf \otimes f) = \begin{pmatrix} \eta_C \\ u_{SC} \end{pmatrix}; Sf \oplus (Sf \otimes f) = \begin{pmatrix} \eta_C; Sf \\ u_{SC}; Sf \otimes f \end{pmatrix} = f; \begin{pmatrix} \eta_D \\ u_D \end{pmatrix}$$

and so naturality of d is established.

To show that d adheres to the chain rule, it is necessary and sufficient to show that the following square commutes

$$
\begin{array}{ccc}
S^2C & \xrightarrow{\quad\quad\quad\mu_C\quad\quad\quad} & SC \\
{\scriptstyle d_{S^2C}}\downarrow & & \downarrow{\scriptstyle d_{SC}} \\
S^2C \otimes SC & \xrightarrow[\mu_C \otimes d_{SC}]{} SC \otimes SC \otimes C \xrightarrow[m_{SC} \otimes 1]{} & SC \otimes C
\end{array}
$$

When preceded by η_{SC}, commutativity of the resultant diagram is established by a routine verification. In order to show that this verification suffices, it must be shown that both paths in the above diagram yield derivations when preceded by η_{SC}; the correspondence between derivations and morphisms of algebras then enables the utilization of the universal property of η to deduce that the associated morphisms of algebras are equal.

Since μ_C is an associative algebra homomorphism, $\mu_C; d_{SC}$ is a derivation with respect to the S^2C-module structure that $SC \otimes C$ acquires by restriction of scalars along μ_C. As for the

9

counterclockwise composite, the following computation demonstrates that it adheres to the Leibniz rule

$$m_{S^2C}; d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1$$
$$= (1 \otimes d_{S^2C} + c; 1 \otimes d_{S^2C}); m_{S^2C} \otimes 1; \mu_C \otimes d_{SC}; m_{SC} \otimes 1$$
$$= (1 \otimes d_{S^2C} + c; 1 \otimes d_{S^2C}); \mu_C \otimes \mu_C \otimes 1; m_{SC} \otimes 1; 1 \otimes d_{SC}; m_{SC} \otimes 1$$
$$= (1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}) + c; 1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC})); \mu_C \otimes 1 \otimes 1 \otimes 1; m_{SC} \otimes 1 \otimes 1; m_{SC} \otimes 1$$
$$= (1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1) + c; 1 \otimes (d_{S^2C}; \mu_C \otimes d_{SC}; m_{SC} \otimes 1)); \mu_C \otimes 1 \otimes 1; m_{SC} \otimes 1$$

That the counterclockwise composite is 0 when preceded by e_{S^2C} is immediate, and the proof is complete. □

4   Beck T-derivations

We now explore what we consider to be the main contribution of this paper. The first step in this project is the following theorem, due to the second author. It lifts the correspondence between derivations and algebra homomorphisms to the level of T-algebras. Throughout this section, we assume that $\mathcal{C}$ has finite coproducts.
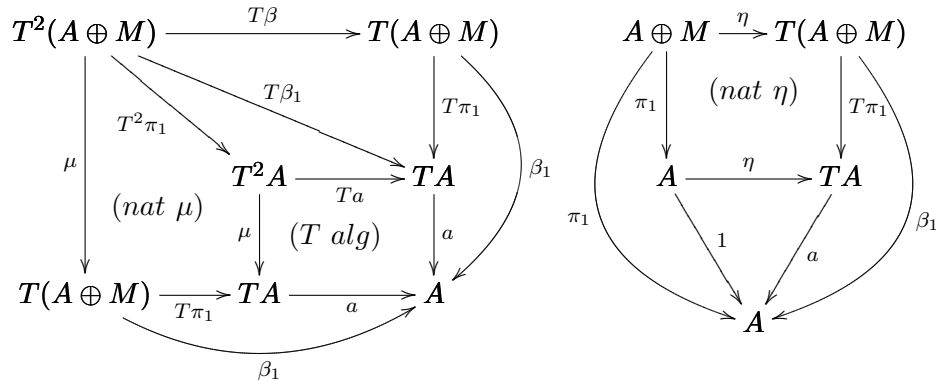
Theorem 4.1. *Let $\mathcal{C}$ be a codifferential category with finite coproducts. Let (A, ν) be a T-algebra and M a module over its associated algebra. Then $(A \oplus M, \beta)$ is a T-algebra with $\beta \colon T(A \oplus M) \to A \oplus M$ defined as follows.*

Evidently we need maps to A and to M which we define as follows:

$$\beta_1 \colon T(A \oplus M) \xrightarrow{T\pi_1} TA \xrightarrow{\nu} A$$

$$\beta_2 \colon T(A \oplus M) \xrightarrow{d} T(A \oplus M) \otimes (A \oplus M) \xrightarrow{T(\pi_1) \otimes \pi_2} T(A) \otimes M \xrightarrow{\nu \otimes 1} A \otimes M \xrightarrow{\bullet} M$$

Proof. The following four diagrams capture all of the necessary equations.

$$
\begin{array}{c}
T^2(A \oplus M) \xrightarrow{\quad T\beta \quad} T(A \oplus M) \\[2pt]
\text{(nat } d\text{)}
\end{array}
$$

d

$T^2(A \oplus M) \otimes T(A \oplus M) \xrightarrow{\ T\beta \otimes \beta\ } T(A \oplus M) \otimes (A \oplus M)$

d

$T\pi_1 \otimes \pi_2$

$T^2\pi_1 \otimes d$ $T\beta_1 \otimes \beta_2$

$T^2 A \otimes T(A \oplus M) \otimes (A \oplus M) \qquad TA \otimes M$

$Ta \otimes T\pi_1 \otimes \pi_2$

(chain rule)

$\mu \qquad \mu \otimes d \qquad (\text{nat } \mu) \qquad \mu \otimes T\pi_1 \otimes \pi_2 \qquad (T \text{ alg}) \qquad (TA)^{\otimes 2} \otimes M \qquad a \otimes 1 \qquad \beta_2$

$a \otimes a \otimes 1$

$(T(A \oplus M))^{\otimes 2} \otimes (A \oplus M) \xrightarrow{T\pi_1 \otimes T\pi_1 \otimes \pi_2} (TA)^{\otimes 2} \otimes M \xrightarrow{a \otimes a \otimes 1} A^{\otimes 2} \otimes M \xrightarrow{1 \otimes \bullet} A \otimes M$

$m_{T(A \oplus M)} \otimes 1 \quad (\text{alg hom}) \quad m_{TA} \otimes 1 \quad (\text{alg hom}) \quad m_A \otimes 1 \quad (\text{act}) \quad \bullet$

$T(A \oplus M) \xrightarrow{d} T(A \oplus M) \otimes (A \oplus M) \xrightarrow{T\pi_1 \otimes \pi_2} TA \otimes M \xrightarrow{a \otimes 1} A \otimes M \xrightarrow{\ \bullet\ } M$

β

β_2

$T(A \oplus M) \xrightarrow{d} T(A \oplus M) \otimes (A \oplus M) \xrightarrow{T\pi_1 \otimes \pi_2} TA \otimes M \xrightarrow{a \otimes 1} A \otimes M \xrightarrow{\ \bullet\ } M$

$\eta \qquad e \otimes 1 \quad (T\pi_1 \text{ alg hom}) \quad e \otimes \pi_2 \quad (a \text{ alg hom}) \quad e \otimes \pi_2 \qquad \pi_2$

$A \oplus M \xrightarrow{\cong} k \otimes (A \oplus M) \xrightarrow{1} k \otimes (A \oplus M) \xrightarrow{1} k \otimes (A \oplus M) \xrightarrow{\cong} A \oplus M$

1

$\square$

Definition 4.2. We denote this T-algebra by $W(A, M) = \langle A \oplus M, \beta^{AM} \rangle$.

The following result is straightforward.

Lemma 4.3. *Let (A, ν) be a T-algebra. Let M an A-module. Then $\pi_1 \colon A \oplus M \to A$ is a map of T-algebras, where $A \oplus M$ is given the T-algebra structure just defined.*

We also note that the algebra associated to this T-algebra under the process of Theorem 2.12 coincides with the algebra structure associated to $A \oplus M$ in Remark 2.6.

Proposition 4.4. *Let (A, α) be a T-algebra in $\mathcal{C}$ and let M be an A-module. Then the commutative algebra structure carried by the T-algebra $A \oplus M$ coincides with the commutative algebra structure on $A \oplus M$ described in Remark 2.6.*

Proof. Since β^{AM} is an algebra homomorphism the multiplication associated to $W(A, M)$ is

$$m_{W(A,M)} = \eta_{A \oplus M} \otimes \eta_{A \oplus M}; m_{T(A \oplus M)}; \beta^{AM}$$

Since $\pi_1 : W(A, M) \longrightarrow A$ is a T-homomorphism and hence an algebra homomorphism, $m_{W(A,M)}; \pi_1 = \pi_1 \otimes \pi_1; m_A$ and so the first component of $m_{W(A,M)}$ is given as in Remark 2.6.

The second component is the composite

$$\eta \otimes \eta; m_{T(A \oplus M)}; d_{T(A \oplus M)}; T\pi_1 \otimes \pi_2; \alpha \otimes 1; \bullet$$

Calculate as follows:

$$
\begin{aligned}
&\eta_{A \oplus M} \otimes \eta_{A \oplus M}; m_{T(A \oplus M)}; d_{T(A \oplus M)}; T\pi_1 \otimes \pi_2; \alpha \otimes 1; \bullet \\
&= \eta_{A \oplus M} \otimes \eta_{A \oplus M}; (1 \otimes d_{T(A \oplus M)} + c; 1 \otimes d_{T(A \oplus M)}); m_{T(A \oplus M)} \otimes 1; S\pi_1 \otimes \pi_2; \alpha \otimes 1; \bullet \\
&= (\eta_{A \oplus M} \otimes (\eta_{A \oplus M}; d_{T(A \oplus M)}) + c; \eta_{A \oplus M} \otimes (\eta_{A \oplus M}; d_{A \oplus M})); (T\pi_1; \alpha) \otimes (T\pi_1; \alpha) \otimes \pi_2; m_A \otimes 1; \bullet \\
&= (1 \otimes (\eta_{A \oplus M}; d_{A \oplus M}) + c; 1 \otimes (\eta_{A \oplus M}; d_{A \oplus M})); (\pi_1; \eta_A; \alpha) \otimes (T\pi_1; \alpha) \otimes \pi_2; m_A \otimes 1; \bullet \\
&= (1 + c); 1 \otimes e_{A \oplus M} \otimes 1; \pi_1 \otimes (T\pi_1; \alpha) \otimes \pi_2; m_A \otimes 1; \bullet \\
&= (1 + c); \pi_1 \otimes \pi_2; 1 \otimes e_A \otimes 1; m_A \otimes 1; \bullet \\
&= (1 + c); \pi_1 \otimes \pi_2; \bullet
\end{aligned}
$$

$\square$

We will need the following technical lemmas concerning the T-algebra $W(A, M)$.

Lemma 4.5. *Let (A, a) be a T-algebra, and let M and N be A-modules. Suppose $h \colon M \to N$ is an A-module map. Then $A \oplus h \colon A \oplus M \to A \oplus N$ is a T-algebra map $W(A, M) \to W(A, N)$.*

Proof. The result follows from the commutativity of the following two diagrams.

$$
\begin{array}{ccc}
T(A \oplus M) & \xrightarrow{\;T(1_A\oplus h)\;} & T(A \oplus N) \\[2pt]
\Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d} \\[2pt]
T(A\oplus M)\otimes(A\oplus M) & \xrightarrow{\;T(1_A\oplus h)\otimes(1_A\oplus h)\;} & T(A\oplus N)\otimes(A\oplus N) \\[2pt]
\Big\downarrow{\scriptstyle T\pi_1\otimes\pi_2} & & \Big\downarrow{\scriptstyle T\pi_1\otimes\pi_2} \\[2pt]
TA\otimes M & \xrightarrow{\;1_{TA}\otimes h\;} & TA\otimes N \\[2pt]
\Big\downarrow{\scriptstyle a\otimes 1_M} & & \Big\downarrow{\scriptstyle a\otimes 1_N} \\[2pt]
A\otimes M & \xrightarrow{\;1_A\otimes h\;} & A\otimes N \\[2pt]
\Big\downarrow{\scriptstyle \bullet_M} & & \Big\downarrow{\scriptstyle \bullet_N} \\[2pt]
M & \xrightarrow{\;\;h\;\;} & N
\end{array}
$$

with outer curves β_2^{AM} (left) from $T(A\oplus M)$ to M, and β_2^{AN} (right) from $T(A\oplus N)$ to N. $\qquad\square$

The above calculations allow us to conclude:

Proposition 4.6. *Given a T-algebra A, the above construction defines a functor:*

$$W(A,-)\colon A\text{-}Mod \longrightarrow \mathcal{C}^T/A$$

Here, $\mathcal{C}^T$ is the category of T-algebras and $\mathcal{C}^T/A$ is the slice category over A.

It is the above series of observations that allows us to define a generalized notion of derivation depending on the given codifferential structure of $\mathcal{C}$.

Definition 4.7.
- Let (A,ν) be a T-algebra. Let M be an A-module. A *Beck T-derivation* for A valued in M is a T-algebra map

$$A \longrightarrow W(A,M) \quad \text{in } \mathcal{C}^T/A$$

  in the slice category $\mathcal{C}^T/A$.

- A *T-derivation* is a morphism $\partial\colon A \to M$ such that

$$\langle 1, \partial\rangle\colon A \longrightarrow A\oplus M$$

  is a T-algebra homomorphism $A \to W(A,M)$.

Remark 4.8. Under the assumptions of Remark 3.1, suppose we are given $A \in \mathcal{C}^S$ where S is the symmetric algebra monad and $M \in A - Mod$. Then a morphism $\partial\colon A \longrightarrow M$ in $\mathcal{C}$ is an S-derivation if and only if ∂ is a derivation.

Remark 4.9. Evidently, the two notions of *Beck T-derivation* and *T-derivation* are in bijective correspondence and we will use the two interchangeably.

We now give several equations for a map $\partial\colon A \to M$ which are equivalent to ∂ being a T-derivation.

Proposition 4.10. *Let (A, ν) be a T-algebra, and let M be an A-module. A morphism $\partial\colon A \to M$ is a T-derivation if and only if the following diagram commutes.*

$$
\begin{array}{ccc}
TA & \xrightarrow{\ T\binom{1_A}{\partial}\ } & T(A \oplus M) \\
{\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \beta_2} \\
A & \xrightarrow{\quad \partial \quad} & M
\end{array}
$$

Proof. Since $A \oplus M$ is a product, the requirement that $\langle 1_A, \partial \rangle\colon A \to A \oplus M$ be a T-algebra homomorphism amounts to two equations, the second of which is expressed by the above diagram whereas the first commutes by the following calculation

$$
\begin{array}{ccc}
TA & \xrightarrow{\ T\binom{1_A}{\partial}\ } & T(A \oplus M) \\
{\scriptstyle \nu}\big\downarrow \quad {\scriptstyle 1_{TA}}\searrow & & \big\downarrow{\scriptstyle T\pi_1} \\
& & TA \\
& & \big\downarrow{\scriptstyle \nu} \\
A & \xrightarrow{\quad 1_A \quad} & A
\end{array}
$$

$\square$

Proposition 4.11. *Let (A, ν) be a T-algebra, and let M be an A-module. A morphism $\partial\colon A \to M$ is a T-derivation if and only if*

$$
\begin{array}{ccc}
TA & \xrightarrow{\quad \nu \quad} & A \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \partial} \\
TA \otimes A & \xrightarrow[\nu \otimes \partial]{} A \otimes M \xrightarrow[\bullet_M]{} & M
\end{array}
$$

commutes.

Proof. Calculate as follows:

$$
\begin{array}{ccc}
TA & \xrightarrow{\quad d \quad} & TA \otimes A \\
{\scriptstyle T\binom{1_A}{\partial}}\big\downarrow & {\scriptstyle T\binom{1_A}{\partial} \otimes \binom{1_A}{\partial}}\big\downarrow & \\
T(A \oplus M) & \xrightarrow[d]{} T(A \oplus M) \otimes (A \oplus M) \xrightarrow[T\pi_1 \otimes \pi_2]{} TA \otimes M \xrightarrow[\nu \otimes 1_M]{} A \otimes M \xrightarrow{\bullet_M} M
\end{array}
$$

with diagonal arrows $\nu \otimes \partial$, $1_{TA} \otimes \partial$, and β_2.

Thus the result follows from the previous proposition.

$\square$

Whereas we have defined the notion of T-derivation in the setting of a given codifferential category, Theorem 4.11 furnishes an equivalent definition that is applicable more generally, as follows.

Definition 4.12. Let $\mathcal{C}$ be a symmetric monoidal category equipped with an algebra modality T and arbitrary morphisms $d_{TC} : TC \to TC \otimes C$ ($C \in \mathcal{C}$). Given a T-algebra A and an A-module M, a *T-derivation* is a morphism $\partial : A \to M$ such that the diagram of Proposition 4.11 commutes.

The new understanding of derivations captured by the above propositions allows us, among other things, to reexamine the definition of (co)differential categories, as seen by the following:

Theorem 4.13. *Let $\mathcal{C}$ be a symmetric monoidal category equipped with an algebra modality T and arbitrary morphisms $d_{TC} : TC \to TC \otimes C$ ($C \in \mathcal{C}$). The Chain Rule equation for d in the definition of codifferential category is equivalent to the statement that each component d_{TC} is a T-derivation, where $TC \otimes C$ is viewed as the free TC-module generated by C.*

Proof.

$$
\begin{array}{ccc}
T^2C & \xrightarrow{\hspace{3cm}\mu\hspace{3cm}} & TC \\
{\scriptstyle d_{T^2C}}\big\downarrow & & \big\downarrow{\scriptstyle d_{TC}} \\
T^2C \otimes TC \xrightarrow{\mu \otimes d_{TC}} TC \otimes TC \otimes C \xrightarrow{m_{TC} \otimes 1_C} & & TC \otimes C
\end{array}
$$

This equation is both the chain rule and the statement that d_{TC} is a derivation.

$\square$

4.1 Universal Beck T-derivations

Definition 4.14. Given a T-algebra A, a *module of Kähler T-differentials* is an A-module, denoted Ω_A^T, equipped with a universal T-derivation on A. This can be expressed in either of the following two equivalent ways:

- A T-derivation $d \colon A \to \Omega_A^T$ such that for all T-derivations $\partial \colon A \to M$, there is a unique A-linear map $\hat{\partial} \colon \Omega_A^T \to M$ such that $d; \hat{\partial} = \partial$.

- A morphism $g \colon A \to W(A, \Omega_A^T)$ in $\mathcal{C}^T/A$ such that for each map $\partial \colon A \to W(A, M)$ in $\mathcal{C}^T/A$, there is a unique A-linear homomorphism $\hat{\partial} \colon \Omega_A^T \to M$ such that $g; W(A, \hat{\partial}) = \partial$.

We now explore the existence of universal derivations from this new T-perspective.

Theorem 4.15. *Let $\mathcal{C}$ be a codifferential category, and let C be an object of $\mathcal{C}$. Then $d_{TC} \colon TC \to T(C) \otimes C$ is a universal T-derivation.*

Proof. Since d_{TC} satisfies the chain rule, it is a T-derivation. Since $T(C) \otimes C$ is the free $T(C)$-module on C, given any T-derivation $\partial : T(C) \longrightarrow M$ there exists a unique $T(C)$-linear morphism $\partial^{\#} : TC \otimes C \longrightarrow M$ such that $u_C^{T(C)}; \partial^{\#} = \eta_C; \partial$. Hence by axiom (d3), the two morphisms from C to M in the following diagram are equal:

$$C \xrightarrow{\eta_C} T(C) \xrightarrow{d_{TC}} T(C) \otimes C$$

with ∂ going to M and $\partial^{\#}$ from $T(C) \otimes C$ to M.

Equivalently,

$$TC \xrightarrow{\binom{1}{d_{TC}}} W(TC, TC \otimes C)$$

with $\binom{1}{\partial}$ and $W(TC, \partial^{\#})$ going to $W(TC, M)$.

commutes when preceded by η_C. Since this is a diagram of T-algebra homomorphisms, it commutes if and only if it commutes when preceded by η_C

$\square$

We now address the issue of extending the existence of universal T-derivations to arbitrary T-algebras.

Proposition 4.16. *Let (A, a) and (B, b) be T-algebras and M a B-module. Let $g\colon A \to B$ be a T-algebra homomorphism. Then $g \oplus M\colon A \oplus M \to B \oplus M$ is a map of T-algebras $W(A, M_A) \to W(B, M)$, where M_A is M with evident induced action of A.*

Proof. The result follows from the commutativity of the following two diagrams.

$$T(A \oplus M) \xrightarrow{T(g \oplus 1_M)} T(B \oplus M)$$

with $T\pi_1$ vertical maps, β_1^{AM} and β_1^{BM} outer arcs, $TA \xrightarrow{Tg} TB$, a and b vertical, $A \xrightarrow{g} B$.

$$
\begin{array}{ccc}
T(A\oplus M) & \xrightarrow{\;T(g\oplus 1_M)\;} & T(B\oplus M)\\
\Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d}\\
T(A\oplus M)\otimes(A\oplus M) & \xrightarrow{\;T(g\oplus 1_M)\otimes(g\oplus 1_M)\;} & T(B\oplus M)\otimes(B\oplus M)\\
\Big\downarrow{\scriptstyle T\pi_1\otimes\pi_2} & & \Big\downarrow{\scriptstyle T\pi_1\otimes\pi_2}\\
TA\otimes M & \xrightarrow{\;Tg\otimes 1_M\;} & TB\otimes M\\
\Big\downarrow{\scriptstyle a\otimes 1_M} & & \Big\downarrow{\scriptstyle b\otimes 1_M}\\
A\otimes M & \xrightarrow{\;g\otimes 1_M\;} & B\otimes M\\
\Big\downarrow{\scriptstyle \bullet_M} & & \Big\downarrow{\scriptstyle \bullet_M}\\
M & \xrightarrow{\;1_M\;} & M
\end{array}
$$

with outer arcs β_2^{AM} (left) and β_2^{BM} (right).

$\square$

Proposition 4.17. *With assumptions as in previous proposition, let $\partial\colon A\to M$ be such that $\langle g,\partial\rangle\colon A\to W(B,M)$ is a map of T-algebras. Then $\partial\colon A\to M_A$ is a T-derivation.*

Proof. This follows from the following calculation, which uses that $g\oplus 1_M$ is a T-algebra homomorphism by the previous proposition.

$$
\begin{array}{ccc}
TA & \xrightarrow{\;T\left(\begin{smallmatrix}1_A\\\partial\end{smallmatrix}\right)\;} & T(A\oplus M)\\
\Big\downarrow{\scriptstyle a} & T\left(\begin{smallmatrix}g\\\partial\end{smallmatrix}\right)\searrow\quad T(B\oplus M)\quad\nwarrow T(g\oplus 1_M) & \Big\downarrow{\scriptstyle \beta_2^{AM}}\\
 & \Big\downarrow{\scriptstyle \beta_2^{BM}} & \\
 & M & \\
A & \xrightarrow[\quad\partial\quad]{} & M
\end{array}
$$

with ∂ from A to M and 1_M from M to M.

$\square$

Definition 4.18. Let Alg be the category of commutative algebras in a codifferential category $\mathcal{C}$ and let $(-)-Mod\colon Alg^{op}\to Cat$ be the usual functor associating to an algebra its category of representations. The functor acts on morphisms by the usual restriction of scalars.

Composing with the functor $F^{op}\colon (\mathcal{C}^T)^{op}\to Alg^{op}$ we obtain a functor $H\colon \mathcal{C}^{T\,op}\to Cat$. When we apply the usual Grothendieck construction to this functor, we obtain a category fibred over $\mathcal{C}^T$ which we call Mod_T. Objects are pairs (A,M) with A a T-algebra and M an A-module. Arrows are pairs $(g,h)\colon (A,M)\to (B,N)$ with $g\colon A\to B$ a T-algebra map and $h\colon M\to N_A$ a map of A-modules. Here N_A is the restriction of scalars of N along g (Remark 2.6).

Theorem 4.19. *There is a functor $W\colon Mod_T \to (\mathcal{C}^T)^{\to}$ that makes the following diagram commute:*

$$
\begin{array}{ccc}
Mod_T & \xrightarrow{\;\;W\;\;} & (\mathcal{C}^T)^{\to} \\
& \searrow \qquad \swarrow {\scriptstyle cod} & \\
& \mathcal{C}^T &
\end{array}
$$

The functor is defined by:

$$On\ objects:\quad (A, M) \mapsto [W(A, M) \xrightarrow{\;\pi_1\;} A]$$

$$On\ arrows:\ (A, M) \xrightarrow{\;(g,h)\;} (B, N) \mapsto the\ following:$$

$$
\begin{array}{ccc}
W(A, M) & \xrightarrow{W(h,g):=g\oplus h} & W(B, N) \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
A & \xrightarrow{\quad g \quad} & B
\end{array}
$$

This functor is fibred over the base category $\mathcal{C}^T$.

Proof. We evidently have that $(1 \oplus h); (g \oplus 1) = g \oplus h$ is a map of T-algebras by Lemma 4.5 and Proposition 4.16, and so we have a functor making the triangle commute.

Now given a T-algebra homomorphism $g\colon A \to B$ and a B-module N, we get a cartesian arrow over g in Mod_T as $(g, 1_N)\colon (A, N_A) \to (B, N)$. It suffices to show that

$$
\begin{array}{ccc}
W(A, N_A) & \xrightarrow{\;W(g,1_N)\;} & W(B, N) \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
A & \xrightarrow{\quad g \quad} & B
\end{array}
$$

is a pullback. Given $f\colon Q \to A$ and $q\colon Q \to W(B, N)$ in $\mathcal{C}^T$ such that $f; g = q; \pi_1$, we find that $q = \langle f; g, \partial \rangle$ for some $\partial\colon Q \to N$. By Lemma 4.17, we conclude $\partial\colon Q \to N_Q$ is a T-derivation. So $\langle 1_Q, \partial \rangle$ is a T-algebra map and thus $\langle 1_Q, \partial \rangle; f \oplus 1 = \langle f, \partial \rangle\colon Q \to W(A, N_A)$ is a T-algebra map. The result now follows. $\qquad\square$

Definition 4.20. Let A be a T-algebra and (B, M) in Mod_T. Let $Der(A, (B, M))$ be the set of all pairs (g, ∂) with $g\colon A \to B$ a T-algebra map and $\partial\colon A \to M_A$ a T-derivation.

We now record two related results which are straightforward.

Proposition 4.21. *The operation Der of the previous definition is functorial in both variables and forms part of a natural isomorphism:*

$$\mathcal{C}^T(A, W(B, M)) \cong Der(A, (B, M))$$

18

This result extends to the slice category in a straightforward way.

Proposition 4.22. *Given a T-algebra map $g\colon A \to B$, we have the following natural isomorphism:*

$$\mathcal{C}^T/B(A, W(B, M)) \cong Der(A, M_A)$$

We now present the main result of the section, demonstrating that the construction of Kähler modules for T-algebras lifts to the setting of T-derivations.

Theorem 4.23. *Suppose $\mathcal{C}$ has reflexive coequalizers, and that these are preserved by $\otimes$ in each variable. Then every T-algebra (A, ν) has a universal T-derivation.*

Proof. Let $g : A \longrightarrow B$ be a morphism of T-algebras, and suppose that universal T-derivations $d_A : A \longrightarrow \Omega^T_A$, $d_B : B \longrightarrow \Omega^T_B$ exist. Then there is a unique A-linear morphism Ω^T_g such that

$$
\begin{array}{ccc}
\Omega^T_A & \xrightarrow{\;\Omega^T_g\;} & \Omega^T_B \\[4pt]
{\scriptstyle d_A}\big\uparrow & & \big\uparrow{\scriptstyle d_B} \\[4pt]
A & \xrightarrow{\;g\;} & B
\end{array}
$$

commutes, where Ω^T_B is considered as an A-module by restriction of scalars along g. This follows from the observation that $g; d_B : A \longrightarrow \Omega^T_B$ is a T-derivation.

Lemma 4.24. *Suppose we are given morphisms in the category Alg as follows which constitute a reflexive coequalizer in $\mathcal{C}$*

$$A_1 \underset{g}{\overset{f}{\rightrightarrows}} A_2 \xrightarrow{\;k\;} A_3$$

Let M_i be an A_i-module for $i = 1, 2$, and let $\phi : M_1 \longrightarrow f^*(M_2)$ and $\gamma : M_1 \longrightarrow g^*(M_2)$ be A_1-linear, where $f^*(M_2)$ and $g^*(M_2)$ denote M_2 equipped with the A_1-module structures induced by f and g, respectively. Suppose

$$M_1 \underset{\gamma}{\overset{\phi}{\rightrightarrows}} M_2 \xrightarrow{\;\kappa\;} M_3$$

is a reflexive coequalizer in $\mathcal{C}$. Then there is a unique A_3-module structure on M_3 such that $\kappa : M_2 \longrightarrow k^*(M_3)$ is A_2-linear.

Proof. Since $\otimes$ preserves reflexive coequalizers, the rows and columns of the following diagram are

reflexive coequalizers:

$$
\begin{array}{ccccc}
A_1 \otimes M_1 & \underset{1\otimes\gamma}{\overset{1\otimes\phi}{\rightrightarrows}} & A_1 \otimes M_2 & \overset{1\otimes\kappa}{\longrightarrow} & A_1 \otimes M_3 \\
{\scriptstyle g\otimes 1}\downdownarrows{\scriptstyle f\otimes 1} & & {\scriptstyle g\otimes 1}\downdownarrows{\scriptstyle f\otimes 1} & & {\scriptstyle g\otimes 1}\downdownarrows{\scriptstyle f\otimes 1} \\
A_2 \otimes M_1 & \underset{1\otimes\gamma}{\overset{1\otimes\phi}{\rightrightarrows}} & A_2 \otimes M_2 & \overset{1\otimes\kappa}{\longrightarrow} & A_2 \otimes M_3 \\
{\scriptstyle k\otimes 1}\downarrow & & {\scriptstyle k\otimes 1}\downarrow & & {\scriptstyle k\otimes 1}\downarrow \\
A_3 \otimes M_1 & \underset{1\otimes\gamma}{\overset{1\otimes\phi}{\rightrightarrows}} & A_3 \otimes M_2 & \overset{1\otimes\kappa}{\longrightarrow} & A_3 \otimes M_3
\end{array}
$$

By Johnstone's lemma, Lemma 0.17, p. 4 [16], it follows that the top row of

$$
\begin{array}{ccccc}
A_1 \otimes M_1 & \underset{g\otimes\gamma}{\overset{f\otimes\phi}{\rightrightarrows}} & A_2 \otimes M_2 & \overset{k\otimes\kappa}{\longrightarrow} & A_3 \otimes M_3 \\
{\scriptstyle \bullet_1}\downarrow & & {\scriptstyle \bullet_2}\downarrow & & {\scriptstyle \bullet_3}\dashdownarrow \\
M_1 & \underset{\gamma}{\overset{\phi}{\rightrightarrows}} & M_2 & \overset{\kappa}{\longrightarrow} & M_3
\end{array}
$$

is also a reflexive coequalizer. We have that

$$
\begin{aligned}
f \otimes \phi; \bullet_2; \kappa &= 1_{A_1} \otimes \phi; f \otimes 1_{M_2}; \bullet_2; \kappa \\
&= \bullet_1; \phi; \kappa \\
&= \bullet_1; \gamma; \kappa \\
&= 1_{A_1} \otimes \gamma; g \otimes 1_{M_2}; \bullet_2; \kappa \\
&= g \otimes \gamma; \bullet_2; \kappa
\end{aligned}
$$

It follows that $\bullet_3 : A_3 \otimes M_3 \longrightarrow M_3$ is constructed as the unique map making the right-hand square in the above diagram commute. Hence it suffices to show that $\bullet_3$ is an A_3-module structure map on M_3. Again using Johnstone's Lemma, the top row of the following diagram is a reflexive coequalizer

$$
\begin{array}{ccccc}
A_1 \otimes A_1 \otimes M_1 & \underset{g\otimes g\otimes\gamma}{\overset{f\otimes f\otimes\phi}{\rightrightarrows}} & A_2 \otimes A_2 \otimes M_2 & \overset{k\otimes k\otimes\kappa}{\longrightarrow} & A_3 \otimes A_3 \otimes M_3 \\
{\scriptstyle 1\otimes\bullet_1;\bullet_1 = m_{A_1}\otimes 1;\bullet_1}\downarrow & & {\scriptstyle 1\otimes\bullet_2;\bullet_2 = m_{A_2}\otimes 1;\bullet_2}\downarrow & & \dashdownarrow \\
M_1 & \underset{\gamma}{\overset{\phi}{\rightrightarrows}} & M_2 & \overset{\kappa}{\longrightarrow} & M_3
\end{array}
$$

It follows that there is a unique map $A_3 \otimes A_3 \otimes M_3 \longrightarrow M_3$ making the right-hand square commute. Since both $1_{A_3} \otimes \bullet_3; \bullet_3$ and $m_{A_3} \otimes 1_{M_3}; \bullet_3$ satisfy this, the result follows. $\square$

Continuing with the proof of our theorem, since μ_A and $T\nu$ are T-algebra morphisms, they induce maps Ω_μ^T and $\Omega_{T\nu}^T$ from $\Omega_{T^2A}^T$ to Ω_{TA}^T, which exist by Theorem 4.14. Furthermore, there exists a map $\Omega_{T\eta}^T$ induced by $T\eta$, which splits both of these maps. Consider the following diagram. We define d_A as the unique morphism in $\mathcal{C}$ such that $\nu; d_A = d_{TA}; \Omega_\nu^T$, which exists since ν is the coequalizer of μ and $T\nu$. Here we take $\Omega_\nu^T : \Omega_{TA}^T \longrightarrow \Omega_A^T$ to be the coequalizer.

$$
\begin{array}{ccccc}
\Omega_{T^2A}^T & \overset{\Omega_\mu^T}{\underset{\Omega_{T\nu}^T}{\rightrightarrows}} & \Omega_{TA}^T & \overset{\Omega_\nu^T}{\longrightarrow} & \Omega_A^T \\[2mm]
\uparrow{\scriptstyle d_{T^2A}} & & \uparrow{\scriptstyle d_{TA}} & & \uparrow{\scriptstyle d_A} \\[2mm]
T^2A & \overset{\mu}{\underset{T\nu}{\rightrightarrows}} & TA & \overset{\nu}{\longrightarrow} & A
\end{array}
$$

One readily verifies that the preceding lemma applies so that Ω_A^T is equipped with an A-module structure, which makes Ω_ν^T TA-linear. We find that $d_A = \eta_A; d_{TA}; \Omega_\nu^T$ since $\nu; \eta_A; d_{TA}; \Omega_\nu^T = d_{TA}; \Omega_\nu^T = \nu; d_A$, where the first equation is established through a short computation using the fact that $\nu; \eta_A = \eta_{TA}; T\nu$.

Since

$$
\begin{array}{ccc}
TA & \overset{\nu}{\longrightarrow} & A \\[2mm]
{\scriptstyle \binom{1_{TA}}{d_{TA}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \binom{1_A}{d_A}} \\[2mm]
W(TA, \Omega_{TA}^T) & \overset{W(\nu, \Omega_\nu^T)}{\longrightarrow} & W(A, \Omega_A^T)
\end{array}
$$

commutes, it follows that the right-hand map is a T-algebra homomorphism and therefore that d_A is a T-derivation. Indeed, the counterclockwise composite is evidently a T-algebra homomorphism, and since ν is a T-algebra homomorphism that is split epi in $\mathcal{C}$, the fact that the right-hand map is a T-algebra homomorphism follows readily.

Now suppose that $\partial : A \longrightarrow M$ is a T-derivation. Then $\nu; \partial$ is a T-derivation, which must factor through d_{TA} via a morphism of TA-modules ∂'. Since

$$
\begin{aligned}
d_{T^2A}; \Omega_{T\nu}^T; \partial' &= T\nu; d_{TA}; \partial' \\
&= T\nu; \nu; \partial \\
&= \mu; \nu; \partial \\
&= \mu; d_{TA}; \partial' \\
&= d_{T^2A}; \Omega_\mu^T; \partial'
\end{aligned}
$$

it follows from the universal property of d_{T^2A} that $\Omega_{T\nu}^T; \partial' = \Omega_\mu^T; \partial'$, so that ∂' factors uniquely through Ω_ν^T via a map $\partial^\# : \Omega_A^T \longrightarrow M$. Since $\nu \otimes \Omega_\nu^T$ is a coequalizer, the following computation shows that this map is A-linear:

$$
\begin{aligned}
\nu \otimes \Omega_\nu^T; \bullet_A; \partial^\# &= \bullet_{TA}; \Omega_\nu^T; \partial^\# \\
&= \bullet_{TA}; \partial' \\
&= 1_{TA} \otimes \partial'; \nu \otimes 1_A; \bullet_A \\
&= \nu \otimes \Omega_\nu^T; 1_A \otimes \partial^\#; \bullet_A
\end{aligned}
$$

Finally, we show that $\partial^{\#}$ is the unique A-linear morphism, which makes

$$
\begin{array}{ccc}
A & \xrightarrow{\ d_A\ } & \Omega_A \\
& \searrow{\scriptstyle \partial} & \downarrow{\scriptstyle \partial^{\#}} \\
& & M
\end{array}
$$

commute. First, observe that $\nu; d_A; \partial^{\#} = d_{TA}; \Omega_\nu^T; \partial^{\#} = d_{TA}; \partial' = \nu; \partial$ so that this does indeed commute after cancellation of ν. Now suppose that there exists another A-linear map $k : \Omega_A^T \longrightarrow M$ such that $d_A; k = \partial$. Then

$$
\begin{aligned}
d_{TA}; \Omega_\nu^T; k &= \nu; d_A; k \\
&= \nu; \partial \\
&= d_{TA}; \partial' \\
&= d_{TA}; \Omega_\nu^T; \partial^{\#}
\end{aligned}
$$

The universal property of d_{TA} dictates that $\Omega_\nu^T; k = \Omega_\nu^T; \partial^{\#}$ and therefore $k = \partial^{\#}$ and the proof is complete.

$\square$

5   An alternative definition of (co)differential category

Realization of the importance of the symmetric algebra in the analysis of Kähler categories also has the benefit that it leads to a succinct alternative definition of codifferential category as follows.

Theorem 5.1. *Let $\mathcal{C}$ be an additive symmetric monoidal category for which the symmetric algebra monad S on $\mathcal{C}$ exists. Assume that $\mathcal{C}$ has reflexive coequalizers and that these are preserved by the tensor product in each variable. Then to equip $\mathcal{C}$ with the structure of a codifferential category is, equivalently, to equip $\mathcal{C}$ with*

1. *a monad T,*

2. *a monad morphism $\lambda : \mathsf{S} \to T$, and*

3. *a transformation $d_{TC} : TC \to TC \otimes C$ natural in $C \in \mathcal{C}$*

such that

(a) *the diagram*

$$
\begin{array}{ccc}
SC & \xrightarrow{\ \lambda_C\ } & TC \\
{\scriptstyle d_{SC}}\downarrow & & \downarrow{\scriptstyle d_{TC}} \\
SC \otimes C & \xrightarrow{\lambda_C \otimes 1_C} & TC \otimes C
\end{array}
$$

   commutes for each $C \in \mathcal{C}$, where d_{SC} is the deriving transformation carried by S, and

(b) *the Chain Rule axiom of Definition 2.9 holds, i.e. each d_{TC} is a T-derivation.*

22

Proof. By Remark 3.1, the category of commutative algebras in $\mathcal{C}$ is monadic over $\mathcal{C}$ and so can be identified with the category of S-algebras. By Theorem 3.2, we know that algebra modalities on $\mathcal{C}$ are in bijective correspondence with pairs (T, λ) consisting of a monad T on $\mathcal{C}$ and a monad morphism $\lambda \colon \mathsf{S} \to T$. Suppose we are given such a pair (T, λ), together with a natural tranformation $d_{T(-)}$ satisfying (a) and (b).

> *Claim:* Any T-derivation $\partial : A \to M$ is, in particular, an S-derivation, equivalently by Remark 4.8, a derivation in the ordinary sense (Definition 2.5).

To prove this claim, observe that the following diagram commutes, by (a) and Definition 4.12, where a is the given T-algebra structure on A.

$$
\begin{array}{ccccc}
SA & \xrightarrow{\ \lambda_A\ } & TA & \xrightarrow{\quad a \quad} & A \\
{\scriptstyle d_{SA}}\downarrow & & {\scriptstyle d_{TA}}\downarrow & & \downarrow{\scriptstyle \partial} \\
SA \otimes A & \xrightarrow[\lambda_A \otimes 1]{} & TA \otimes A & \xrightarrow[a \otimes \partial]{} & A \otimes M \xrightarrow{\bullet} M
\end{array}
$$

But the upper row is the S-algebra structure acquired by A via Theorem 3.2, so by Definition 4.12 the Claim is proved.

We have assumed that d satisfies the Chain Rule axiom, equivalently that each component $d_{TC} : TC \to TC \otimes C$ is a T-derivation (Theorem 4.13), so by the Claim, d_{TC} is an S-derivation, equivalently, an ordinary derivation. Hence the axioms (d1) and (d2) of Definition 2.9 hold, since together they assert exactly that each component d_{TC} is an ordinary derivation. We also know that axiom (d4) (the Chain Rule) holds, by assumption (b), so it suffices to prove that (d3) holds. Indeed, (d3) asserts that the periphery of the following diagram commutes

$$
\begin{array}{ccccc}
& & \overset{\eta_C^T}{\xrightarrow{\hspace{5cm}}} & & \\
C & \underset{\eta_C^{\mathsf{S}}}{\xrightarrow{\hspace{1.2cm}}} & SC & \underset{\lambda_C}{\xrightarrow{\hspace{1.2cm}}} & TC \\
{\scriptstyle \iota}\downarrow & & {\scriptstyle d_{SC}}\downarrow & & \downarrow{\scriptstyle d_{TC}} \\
k \otimes C & \xrightarrow{e \otimes 1} & SC \otimes C & \xrightarrow{\lambda_C \otimes 1} & TC \otimes C \\
& & \underset{e \otimes 1}{\xrightarrow{\hspace{4cm}}} & &
\end{array}
\tag{1}
$$

where η^T and η^{S} are the units of T and S, respectively. The upper cell commutes since λ is a monad morphism, and the lower cell commutes since λ_C is an S-homomorphism, i.e a homomorphism of algebras. The leftmost cell commutes since $\mathcal{C}$ is a codifferential category when equipped with S (3.4), and the rightmost cell commutes by (a).

Conversely, let us instead assume that $(\mathcal{C}, T, d)$ is a codifferential category. Then since axiom (d3) holds, the periphery of the diagram (1) commutes, but we also know that the upper, lower, and leftmost cells in (1) commute. Hence, whereas our aim is to show that (a) holds, i.e., that the rightmost square in (1) commutes, we know that this square 'commutes when preceded by η_C^{S}'. But by axioms (d1) and (d2), d_{TC} is an ordinary derivation, equivalently, an S-derivation (2.5), so the composite $\lambda_C; d_{TC}$ is an S-derivation since λ_C is an algebra map. Also, d_{SC} is an S-derivation, and one readily checks that $\lambda_C \otimes 1 : SC \otimes C \to TC \otimes C$ is a morphism of SC-modules (where $TC \otimes C$ carries the SC-module structure that it acquires by restriction of scalars along the

algebra homomorphism λ_C). Hence the composite $d_{SC}; \lambda_C \otimes 1$ is an S-derivation. Therefore both composites in the square in question are S-derivations and so are uniquely determined by their composites with $\eta_C^{\mathsf{S}} : C \to SC$, which are equal. $\qquad\square$

An advantage of this definition is that it immediately paves the way for variations of the theory of differential categories and differential linear logic. For example, to obtain noncommutative variants, one can replace the symmetric algebra in the above construction with a different endofunctor.

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